IB Math HL2: Vectors warm up

Name: ____________ period: ____

- Three Dimensional Vector (in space)

Component form of a vector: $\vec{A} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Basis form of a vector:

$\vec{A} = ai + bj + ck$.

1) Example of vector operation:

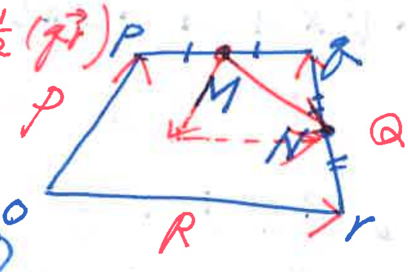
If $\vec{p} = 3i - j + 4k$ and $\vec{q} = -2i + 3k \Rightarrow \frac{3}{2}\vec{p} - \vec{q} = \boxed{\frac{13}{2}i - \frac{3}{2}j + 3k}$

2) Example: Given $\overrightarrow{op} = P$, $\overrightarrow{or} = R$, and $\overrightarrow{rq} = Q$. M is the midpoint of ~~PQ~~ pq and N is ~~th emidpooint~~ the midpoint of ~~QR~~ qr.
Find the following vectors in terms of P, Q, and R.

a) ~~OQ~~ $\vec{oq} = \boxed{R+Q}$ b) ~~PQ~~ $\vec{pq} = \boxed{-P+R+Q}$ c) MN $= \frac{1}{2}(\vec{pq}) + \frac{1}{2}(\vec{qr})$

$= \frac{1}{2}(-P+R+Q)$ $+ \frac{1}{2}(-Q)$

$= -\frac{1}{2}P + \frac{1}{2}R$



- The length of a vector: $\vec{A} = ai + bj + ck$

$|\vec{A}| = \sqrt{a^2+b^2+c^2}$

- Unit vector: $\frac{\vec{A}}{|\vec{A}|} = \frac{ai+bj+ck}{\sqrt{a^2+b^2+c^2}}$

3) Example: Find the vector with a length 5 units and in the direction of . $-i + 2j - 5k$.

$\frac{-5i}{\sqrt{30}} + \frac{10j}{\sqrt{30}} - \frac{25k}{\sqrt{30}}$.

- The scalar Product (Dot Product)

$\vec{a} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ $\Rightarrow \vec{a} \cdot \vec{b} = (a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2)$

$\vec{b} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ $\cos\theta = \frac{a \cdot b}{|a||b|}$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

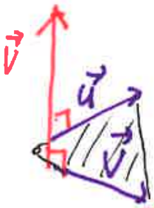
4) Example: Find the angle between the two vectors $\vec{a} = 3i - j + 4k$ and $\vec{b} = -2i + 3k$.

$\theta = 71.0°$

- The cross Project (Vector Product)

$\vec{a} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i\,(b_1 c_2 - c_1 b_2) - j\,(a_1 c_2 - c_1 a_2) + k\,(a_1 b_2 - b_1 a_2)$$



The properties of Cross Product: given vector $\vec{u}$ and $\vec{v}$.

- $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$.
- $\frac{1}{2}|\vec{u} \times \vec{v}| = \frac{1}{2}|\vec{u}||\vec{v}| \sin\theta$: Area of $\triangle$ formed by $\vec{u}$ and $\vec{v}$.
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times \vec{u} = 0$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

5) Example) Given $\vec{a} = 3i - j + 4k$ and $\vec{b} = 3i + j - 2k$

a) Find $\vec{a} \times \vec{b}$

$= \boxed{-2i + 18j + 6k}$

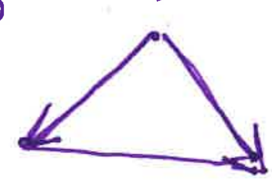
$= \begin{pmatrix} -2 \\ 18 \\ 6 \end{pmatrix}$

b) Find $\vec{b} \times \vec{a}$

$= \boxed{2i - 18j - 6k}$

c) $\vec{a} \times \vec{a}$

$= 0$



6) Example:

a) Find a perpendicular vector to a plane formed by A (1, 2,3), B(0, -3,5), and C(-3,1,-1).

$\vec{AB} = \begin{pmatrix} 0 & -1 \\ -3 & -2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} -3 & -1 \\ 1 & -2 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ -4 \end{pmatrix}$

$\vec{AB} \times \vec{AC} = 22i - 12j - 19k$

Hence,

b) Find the area of the triangle formed by A, B, and C.

$$\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{(22)^2 + (12)^2 + (19)^2}$$

$$= \frac{1}{2}\sqrt{989}$$

Practice)

1. Find a vector perpendicular to both $\vec{v} = 3i - j + k$ and $\vec{w} = 2i + j + k$ and a magnitude of 4 units.

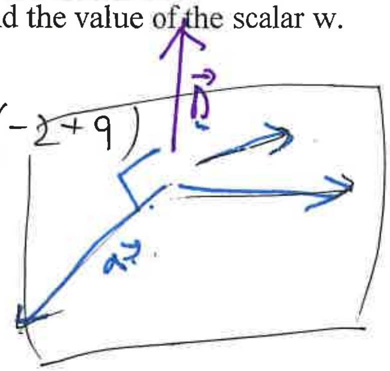
$$\perp \text{ vector}: \vec{v} \times \vec{w} \Rightarrow \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = i(-2) - j(3-2) + k(3+2) = -2i - j + 5k$$

$$\Rightarrow 4\left(\frac{-2i - 2j + 5k}{\sqrt{4+4+25}}\right) = \boxed{\frac{-8}{\sqrt{30}}i - \frac{4j}{\sqrt{30}} + \frac{20k}{\sqrt{30}}}$$

$\vec{a} = 2i + j + wk$

2. If a vectors $\vec{a} = \cancel{i - 3j + wk}$, $\vec{b} = i - 3j + 4k$, and $\vec{c} = 3i - 2j + k$ are coplanar, find the value of the scalar w.

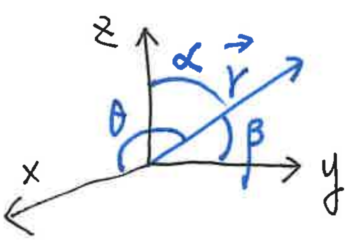
$$\vec{b} \times \vec{c} \Rightarrow \begin{vmatrix} i & j & k \\ 1 & -3 & 4 \\ 3 & -2 & 1 \end{vmatrix} = i(-3+8) - j(1-12) + k(-2+9) = 5i + 11j + 7k = \vec{n}$$



$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0 \Rightarrow \begin{pmatrix} 2 \\ 1 \\ w \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 11 \\ 7 \end{pmatrix} = 10 + 11 + 7w = 0$$

$$7w = -21 \Rightarrow w = -3$$

3. Given vector $\vec{r} = \sqrt{2}i + j - k$, find the angles which r makes with each of x, y, and z axes.



X axis : i

Y axis : j

Z axis : k.

$$x\text{-axis}: \cos\theta = \frac{(\sqrt{2}i + j - k)\cdot i}{\sqrt{2+1+1}\cdot\sqrt{1}} = \frac{\sqrt{2}}{\sqrt{4}} \Rightarrow \theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = 45°$$

$$y\text{-axis}: \cos\beta = \frac{(\sqrt{2}i + j - k)\cdot j}{\sqrt{4}\cdot 1} = \frac{1}{2} \Rightarrow \beta = \arccos\left(\frac{1}{2}\right) = 60°$$

$$z\text{-axis}: \cos\alpha = \frac{(\sqrt{2}i + j - k)\cdot k}{2} = \frac{-1}{2} \Rightarrow \alpha = \arccos\left(\frac{-1}{2}\right) = 120°$$