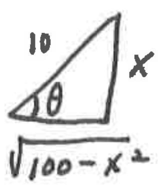
#1 Trig Sub Notes.

a. Start with $x = 10\sin\theta$.

$dx = 10\cos\theta\, d\theta$

$10\cos\theta = \sqrt{100-x^2}$



- $\int \sqrt{100-x^2}\,dx$

$= \int 10\cos\theta \; 10\cos\theta\, d\theta$

$= 100 \int \cos^2\theta\, d\theta$

$= 100\,\frac{1}{2} \int (1+\cos 2\theta)\,d\theta$

$= 50 \int (1+\cos 2\theta)\,d\theta$

$= 50\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

$= 50\left[\theta + \cancel{\left(\frac{1}{2}\right)}(\cancel{2}\sin\theta\cos\theta)\right) + C$

$= 50 \arcsin\left(\frac{x}{2}\right) + \cancel{50}\left(\frac{x}{\cancel{10}}\right)\left(\frac{\sqrt{100-x^2}}{\cancel{10}\;2}\right) + C$

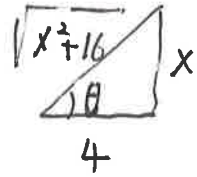
$= 50 \arcsin\left(\frac{x}{2}\right) + \left(\frac{x}{\cancel{5}}\right)\left(\frac{\sqrt{100-x^2}}{1}\right) + C$

$= \boxed{50 \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{100-x^2}}{2} + C}$

b. $\int \sqrt{x^2+16}\,dx$.

$x = 4\tan\theta$.

$dx = 4\sec^2\theta\,d\theta$.



$4\sec\theta = \sqrt{x^2+16}$

$$\int 4\sec\theta \cdot 4\sec^2\theta\,d\theta$$

$$= 16\int \sec^3\theta\,d\theta.$$

$u = \sec\theta$ | $dv = \sec^2\theta\,d\theta$

$du = \sec\theta\cdot\tan\theta\,d\theta$ | $v = \tan\theta$.

$$= 16\left[\tan\theta\sec\theta - \int \sec\theta\tan^2\theta\,d\theta\right]$$

$$= 16\left[\sec\theta\tan\theta - \int \sec\theta\,(\sec^2\theta - 1)\,d\theta\right].$$

$$= \cancel{16}\left[\sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta\right] = \cancel{16}\int \sec^3\theta\,d\theta \quad \therefore$$

$$\Rightarrow \frac{\cancel{2}}{\cancel{2}}\boxed{\int \sec^3\theta\,d\theta} = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|.$$

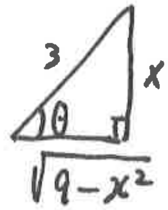
$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$\therefore \int \frac{\sec x\,dx}{1}\,\frac{(\sec x + \tan x)}{(\sec x + \tan x)} = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$

$$= \ln|\sec x + \tan x| + C$$

$$16\left[\frac{1}{2}\,\frac{\sqrt{x^2+16}}{4}\cdot\frac{x}{4} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right|\right] + C$$

c. $\int x^2\sqrt{9-x^2}\,dx$.



Start with $x = 3\sin\theta$

$$dx = 3\cos\theta\,d\theta$$

$$3\cos\theta = \sqrt{9-x^2}$$

$$\int 9\sin^2\theta\cdot 3\cos\theta\cdot 3\cos\theta\,d\theta$$

$$= 81\int \sin^2\theta\cos^2\theta\,d\theta$$

$$= 81\int \tfrac{1}{2}(1-\cos 2\theta)\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{81}{4}\int (1-\cos^2 2\theta)\,d\theta$$

$$= \frac{81}{4}\int \left[1-\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \frac{81}{4}\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{81}{4}\left(\tfrac{1}{2}\theta-\tfrac{1}{8}\sin 4\theta\right)+C$$

$$= \frac{81}{8}\arcsin\left(\frac{x}{3}\right)-\frac{81}{4}\cdot\frac{1}{8}\cdot 2\sin 2\theta\cos 2\theta+C$$

$$= \frac{81}{8}\arcsin\left(\frac{x}{3}\right)-\frac{81}{4}\cdot\frac{1}{\cancel{8}_{2}}\cdot\cancel{2}\cdot 2\cdot\sin\theta\cos\theta(1-\sin^2\theta)+C$$

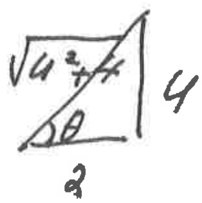
$$= \frac{81}{8}\arcsin\left(\frac{x}{3}\right)-\frac{\cancel{81}^{9}}{8}\cdot\left(\frac{x}{\cancel{3}}\right)\left(\frac{\sqrt{9-x^2}}{\cancel{3}}\right)\left(1-\frac{x^2}{9}\right)+C$$

f. $\int \frac{7}{\sqrt{x^2+2x+5}}\,dx$

$= \int \frac{7}{\sqrt{(x^2+2x+1)+4}}\,dx$

$= \int \frac{7}{\sqrt{(x+1)^2+4}}\,dx$ $\quad u = x+1$, $du = dx$

$= \int \frac{7}{\sqrt{u^2+4}}\,du$ $\quad u = 2\tan\theta$, $du = 2\sec^2\theta\,d\theta$



$= \int \frac{7}{\cancel{2}\sec\theta}\cdot\cancel{2}\sec^{\cancel{2}}\theta\,d\theta$ $\quad 2\sec\theta = \sqrt{u^2+4}$

$= \int 7\sec\theta\,d\theta = \boxed{7\ln|\sec\theta+\tan\theta|+C}$

$= \boxed{7\ln\left|\frac{\sqrt{x^2+2x+5}}{2}+\frac{x+1}{2}\right|+C}$