IB Questions WS (No cal)

Solutions

#1. $\arctan x^2 + \arctan y^2 = \frac{\pi}{4}$

a) $$\frac{2x}{1+(x^2)^2} + \frac{2y \cdot y'}{1+(y^2)^2} = 0$$

$$\boxed{y' = \frac{dy}{dx} = -\left(\frac{1+y^4}{1+x^4}\right)\cdot\frac{x}{y}}$$

b) $x = \frac{1}{\sqrt{2}} \Rightarrow \tan\left[\arctan\left(\frac{1}{2}\right) + \arctan\left(y^2\right)\right] = \tan\left[\frac{\pi}{4}\right]$

$$\tan\left[\arctan\left(\tfrac{1}{2}\right) + \arctan(y^2)\right] = \tan\frac{\pi}{4} = 1$$

Identity

$$\Rightarrow \frac{\cancel{\tan}(\cancel{\arctan}\tfrac{1}{2}) + \cancel{\tan}(\arctan y^2)}{1 - \tan(\arctan\tfrac{1}{2})\cdot\tan(\arctan y^2)}$$

$$= \frac{\frac{1}{2} + y^2}{1 - \frac{1}{2}y^2} = 1 \Rightarrow \frac{1}{2} + y^2 = 1 - \frac{1}{2}y^2$$

$$\frac{3}{2}y^2 = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{3}} \ (y<0)$$

$$y = -\frac{1}{\sqrt{3}}$$

1

$$\frac{dy}{dx} = -\left[\frac{1+\frac{1}{9}}{1+\frac{1}{4}}\right]\cdot\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{3}}}$$

$$= \boxed{\frac{8\sqrt{3}}{9\sqrt{2}}} \text{ OR } \boxed{\frac{4\sqrt{3}}{9}}$$

#2

a) $\cos 2\theta = 2\cos^2\theta - 1$

$\cos x = 2\cos^2 \frac{1}{2}x - 1$

$\cos^2 \frac{1}{2}x = \frac{1}{2}(1+\cos x)$

$\cos\frac{1}{2}x = \pm\sqrt{\frac{(1+\cos x)}{2}} \qquad 0 \le x \le \pi$

$\Rightarrow \boxed{\therefore \cos\left(\frac{1}{2}x\right) = \sqrt{\frac{1+\cos x}{2}} \qquad 0 \le \frac{x}{2} \le \frac{\pi}{2}.}$

b) $\cos x = 1 - 2\sin^2\frac{1}{2}x \quad \Rightarrow \sin^2\frac{1}{2}x = \frac{1}{2}(1-\cos x)$

$\sin\frac{1}{2}x = \pm\sqrt{\frac{1-\cos x}{2}} \qquad 0 \le x \le \pi$

$\therefore \sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}} \qquad 0 \le \frac{x}{2} \le \frac{\pi}{2}$

c)

Hence $\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x} + \sqrt{1-\cos x}\right) dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{2}\left[\cos\left(\frac{1}{2}x\right) + \sin\left(\frac{1}{2}x\right)\right] dx.$

$= 2\sqrt{2}\left[\sin\frac{x}{2} - \cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$= 2\sqrt{2}\left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4} - 0 + 1\right] = \boxed{2\sqrt{2}}$

3

$$f(x) = \frac{\ln x}{x}$$

a. $$f'(x) = \frac{\frac{1}{x}\cdot x - (\ln x)}{x^2} = \boxed{\frac{1-\ln x}{x^2}}$$

b. $$f'(x) = \frac{1-\ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0$$

$$\ln x = 1$$

$$x = e$$

Max $\boxed{\left(e, \frac{1}{e}\right)}$

$f'(x)$ ⊕ e ⊖

c. $$f''(x) = \frac{\left(-\frac{1}{x}\right)(x^2) - (1-\ln x)\cdot 2x}{x^4} = \frac{-x - 2x + 2x\ln x}{x^4} = \frac{-3\cancel{x} + 2\cancel{x}\ln x}{x^{\cancel{4}\,3}}$$

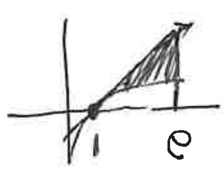
$$= \frac{-3+2\ln x}{x^3} = 0 \Rightarrow \ln x = \frac{3}{2} \Rightarrow x = e^{\frac{3}{2}}.$$

$$f\left(e^{\frac{3}{2}}\right) = \frac{\ln\left(e^{\frac{3}{2}}\right)}{e^{\frac{3}{2}}} = \frac{\frac{3}{2}}{e^{\frac{3}{2}}} = \frac{3}{2e^{\frac{3}{2}}}$$

point of Inflexion : $\left(e^{\frac{3}{2}}, \frac{3}{2e^{3/2}}\right)$:

d. $$f(x) = \frac{\ln x}{x} = 0 \qquad x = 1,\ y = 0.$$

The Equation of the tangent : $\boxed{y = x - 1}$

e. 

$$\Rightarrow \int_1^e \left[(x-1) - \frac{\ln x}{x}\right]dx = \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^e = \boxed{\frac{e^2}{2} - e - \frac{1}{2}}$$