Modeling Situations with Differential Equations

Name: ____________
Period: _________

Punctured Tire Problem

You run over a nail. As the air leaks out of your tire, the rate of change of air pressure inside the tire is directly proportional to that pressure.

k

p: pressure (psi)
t: time (min)

a) Write a differential equation that models this situation.

$$\frac{dp}{dt} = k \cdot p$$

b) Evaluate the proportionality constant if the pressure was 35 psi and decreasing at 0.28 psi/min at time zero.,

$$\frac{dp}{dt} = \frac{-0.28}{35} = \frac{k(35)}{35} \qquad k = \frac{-0.28}{35} \approx \boxed{-.008}$$

c) Solve the differential equation subject to the initial condition implied in step b.

$t=0 \quad p = 35$ psi.

$$\frac{dp}{dt} = -.008p$$

$$\Rightarrow \int \frac{dp}{p} = \int .008\,dt. \Rightarrow \ln p = -.008t + C$$

$$p = e^{-.008t+c}$$

$$p = e^{-.008t} \cdot \boxed{e^c}$$

$$e^c = P_0$$

$$\boxed{p = P_0 \cdot e^{-.008t}}$$

$$p = 35 \cdot e^{-.008t}$$

✓ d) What will the pressure be at 10 min after the tire was punctured?

$p \approx 32.31$ psi.

✓ e) The car is safe to drive as long as the tire pressure is 12 psi or greater. For how long after the puncture will the car be safe to drive?

$t \approx 133.8$ min.

Bacteria Problem

Bacteria in a lab culture grow in such a way that the instantaneous rate of change of bacteria is directly proportional to the number of bacteria present.

t = time (hrs)
B = #s of bacteria (millions).

a) Write a differential equation that expresses the relationship. Separate the variables and integrate the equation, solving the number of bacteria as a function of time.

$$\frac{dB}{dt} = kB \Rightarrow \int \frac{dB}{B} = \int k\,dt$$

$$\ln B = kt + C \Rightarrow B = e^{kt+c} = e^{kt} \cdot \boxed{e^c}, \quad e^c = B_0$$

$$\Rightarrow \boxed{B = B_0 e^{kt}}$$

$t=0$ $\quad B = 5$ million $\quad t=3 \quad B=7$ million

b) Suppose that initially there are 5 million bacteria. Three hours later, the number has grown to 7 million bacteria. Write the particular equation that expresses the number of millions of bacteria as a function of the number of hours.

$B = B_0 e^{kt} \quad t=0 \quad 5 = B_0 \Rightarrow B = 5e^{kt}$

$7 = 5e^{k(3)} \Rightarrow \frac{7}{5} = e^{3k} \Rightarrow \ln \frac{7}{5} = 3k \quad k = \frac{\ln(7/5)}{3}$

$k = .112$

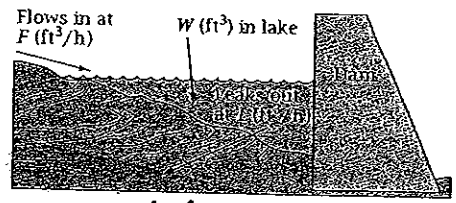
c) What will the bacteria population be one full day after the first measurement?

$B = 73.5$ million.

d) When will the population reach 1 billion ?

$t = 47.3$ hrs.

Dam Leakage Problem



A new dam is constructed across Scorpion Cluch(Figure). The engineers need to predict the volume of water in the lake formed by the dam as a function of time. At time t=0 days, the water starts flowing in a fixed ~~arte~~ rate F, in cubic ft/h. Unfortunately, as the water level rises, some leaks out. The Leakage rate, L, in cubic ft/h, is directly proportional to the volume of water, W, in cubic ft, present in the lake. Thus, the instantaneous rate of change of W is equal to F-L.

kW

$\frac{dW}{dt}$.

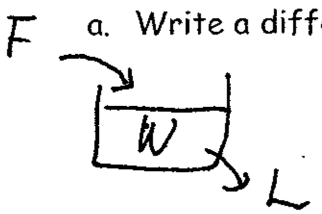
t: time hrs

W: (ft³) water in the lake

F: Flow rate (ft³/hr)

L: Leakage (ft³/hr)

a. Write a differential equation that expresses dW/dt in terms of F, W, and t.



$$\frac{dW}{dt} = F - L \qquad L = kW$$
$$= F - kW$$

b. Solve for W in terms of t, using the initial condition W=0 when t=0.

$$\frac{dW}{F-kW} = dt \Rightarrow \int \frac{dW}{F-kW} = \int dt.$$

c. The engineers know that water is flowing in at $F = 5000\, ft^3/h$. Based on geological considerations, the proportionality constant in the leakage equation is assumed to be 0.04/h. Write the equation for W, substituting the quantities.

$k = 0.04$

$F = 5000$

d. Predict the volume of water in the lake after 10 h, 20 h, and 30 h. After these intervals, how much water has flowed in and how much has leaked out?

$W = \frac{1}{k}(F - C_1 e^{-kt}) \Rightarrow t = 0,\ W = 0 \qquad F = 5000 \qquad k = 0.04$

$\Rightarrow W = 25(5000 - C_1 e^{-0.04t}) \Rightarrow 0 = 25(5000 - C_1 e^0)$

$C_1 = 5000$

$\Rightarrow W = 125000(1 - e^{-0.04t}) \Rightarrow$

$W = 125{,}000(1 - e^{-0.04t})$

t	W	$F \cdot t$	$L \cdot t$
10	41,210	50,000	16480
20	68,834	100,000	2753[illegible] ≠ 55,060
30	87,351	150,000	3494 ≠ 104820

e. When will there be 100,000 cubic ft of water in the lake?

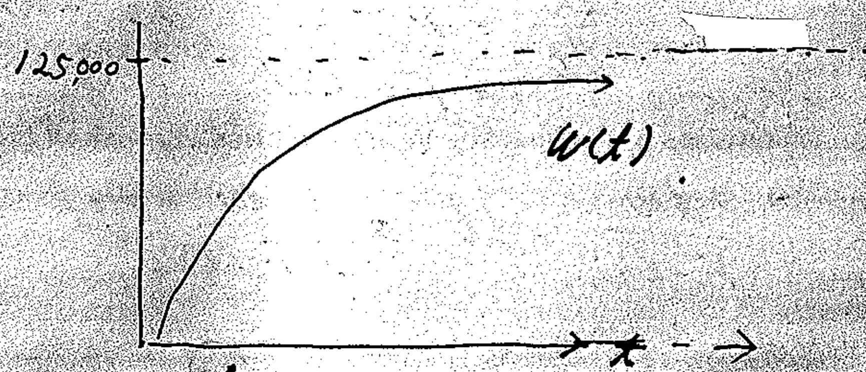
$$100{,}000 = 125000(1 - e^{-0.04t})$$

$$t \approx 40.2359$$

g. Draw the graph of W versus t, showing the asymptote.

$$W = 125000(1 - e^{-0.04t})$$

as $t \to \infty$.



b. $\frac{dw}{dt} = F - kW \Rightarrow \frac{dW}{} = (F - kW)\,dt$

$\Rightarrow \int \frac{dW}{F-kW} = \int dt$

$\Rightarrow \ln(F-kW) \cdot \frac{1}{-k} = t + C$

$\Rightarrow \ln(F-kw) = -k(t+C)$

$W = 0$ when $t = 0$. $\Rightarrow W = \boxed{}$

c. $F = 5000\ ft^3/hr$

$k = 0.04\ /hr$.

$\Rightarrow \ln(5000 - 0.04W) = -0.04(t + C)$

$W = 0 \quad t = 0$.

$\ln(5000) = -0.04\,C$

$C = \frac{\ln(5000)}{-0.04} = \frac{1}{-0.04} \ln(5000)$

$= -25 \ln(5000)$

$\Rightarrow \ln(5000 - 0.04W) = -0.04t + \ln 5000$

$\Rightarrow 5000 - 0.04W = e^{(-0.04t + \ln 5000)}$

$\Rightarrow 5000 - 0.04W = e^{-0.04t} \cdot \cancel{e^{\ln 5000}}$ 5000

$\Rightarrow \cancel{5000} - 0.04W = (5000) \cdot e^{-0.04t}$

$-\cancel{5000} \qquad\qquad -5000$

$\Rightarrow \frac{-0.04W}{-0.04} = \frac{5000 \cdot e^{-0.04t} - 5000}{-0.04}$.

$W = 125{,}000 - 125{,}000\, e^{-0.04t}$

Name: ______________ Group Members: ______________

Exploration 7-2a: Differential Equation for Compound Interest

Date: ______________

Objective: Write and solve a differential equation for the amount of money in a savings account as a function of time.

When money is left in a savings account, it earns interest equal to a certain percent of what is there. The more money you have there, the faster it grows. If the interest is *compounded continuously,* the interest is added to the account the instant it is earned.

1. For continuously compounded interest, the instantaneous rate of change of money is directly proportional to the amount of money. Define variables for time and money, and write a differential equation expressing this fact.

   M: The amount of money at time t

   t: Time (yrs)

   $$\boxed{\frac{dM}{dt} = kM}$$

2. Separate the variables in the differential equation in Problem 1, then integrate both sides with respect to *t*. Transform the integrated equation so that the amount of money is expressed explicitly in terms of time.

   $$\frac{dM}{M} = k\,dt \Rightarrow \int \frac{dM}{M} = \int k\,dt$$

   $$\ln|M| = kt + C \qquad M = e^{kt}\cdot e^{C}$$

3. The integrated equation from Problem 2 will contain *e* raised to a power containing *two* terms. Write this power as a product of two different powers of *e*, one that contains the time variable and one that contains no variable.

   $$M = e^{kt}\cdot A$$

   where $e^{C} = A$

4. You should have the expression e^{C} in your answer to Problem 3. Explain why e^{C} is always positive.

   A positive number raised to any power is positive.

5. Replace e^C with a new constant, C_1. If C_1 is allowed to be positive or negative, explain why you no longer need the ± sign that appeared when you removed the absolute value in Problem 2.

   $M > 0$ Let $A = e^{C}$ positive

   $M < 0$ Let $A = e^{C}$ negative

6. Suppose that the amount of money is $1000 when time equals zero. Use this initial condition to evaluate C_1.

   $$M(0) = A\,e^{k\cdot 0} = 1000$$

7. If the interest rate is 5% per year, then *d*(money)/*d*(time) = 0.05(money) in dollars per year. What, then, does the proportionality constant in Problem 1 equal?

   $k = 0.05$

8. How much money will be in the account after 1 year? 5 years? 10 years? 50 years? 100 years? Do the computations in the most time-efficient manner.

   $M(5) = \$1284.03$ $\quad M(100) \approx \$148,413.16$

   $M(10) \approx \$1648.72$

   $M(50) \approx \$12,182.49$

9. How long would it take for the amount of money to double its initial value?

   $$t = \frac{\ln 2}{0.05} \approx 13.8629 \text{ yr.}$$

   $$2A = A\,e^{kt}$$

10. What did you learn as a result of doing this Exploration that you did not know before?

me: ______________ Group Members: ______________________________

xploration 7-3a: Differential Equation r Memory Retention

Date: __________

bjective: Write and solve a differential equation for the number of names remembered a function of time.

Member is a freshman at a large university. One ening he attends a reception at which there are many embers of his class whom he has not met. He wants to edict how many new names he will remember at the end the reception.

1. Ira assumes that he meets people at a constant rate of R people per hour. Unfortunately, he forgets names at a rate proportional to y, the number he remembers. The more he remembers, the faster he forgets! Let t be the number of hours he has been at the reception. What does dy/dt equal? (Use the letter k for the proportionality constant.)

$$\frac{dy}{dt} = R - ky$$

t: time
R: constant (people he meets)
y: #s of names.

2. The equation in Problem 1 is a differential equation because it has differentials in it. By algebra, separate the variables so that all terms containing y appear on one side of the equation and all terms containing t appear on the other side.

$$\int (R-ky)^{-1}\,dy = \int dt$$

$$-\frac{1}{k}\ln(R-ky) = t + C$$

$$\ln|R-ky| = -kt + C_1$$

3. Integrate both sides of the equation in Problem 2. You should be able to make the integral of the reciprocal function appear on the side containing y.

4. Show that the solution in Problem 3 can be transformed into the form

$$ky = R - Ce^{-kt}$$

where C is a constant related to the constant of integration. Explain what happens to the absolute value sign that you got from integrating the reciprocal function.

$$R - ky = e^{-kt + C_1}$$
$$= e^{-kt} \cdot e^{C_1}$$
$$= e^{-kt} \cdot A$$
$$ky = R - Ae^{-kt}$$

5. Use the initial condition $y = 0$ when $t = 0$ to evaluate the constant C.

$$k \cdot 0 = R - Ae^{0} \Rightarrow A = R$$
$$\Rightarrow y = \frac{R}{k}(1 - e^{-kt})$$

6. Suppose that Ira meets 100 people per hour, and that he forgets at a rate of 4 names per hour when $y = 10$ names. Write the particular equation expressing y in terms of t.

$R = 100$ $\quad ky = 4$ when $y = 10$ $\quad k = \frac{4}{y} = \frac{4}{10}$

$\boxed{y = 250(1 - e^{-0.4t})}$ $\quad k = 0.4$

7. ~~How many~~ ...s will Ira have remembered at the end of the reception, $t = 3$ h?

$y(3) \approx 175$ names. (174)

8. What did you learn as a result of doing this Exploration that you did not know before?