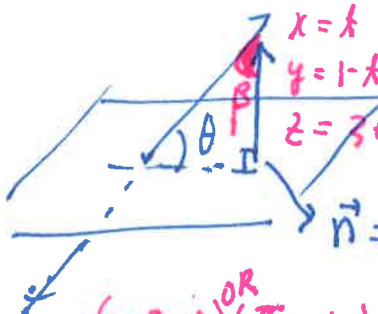
IB Math HL2 Angles in Space Name ________________

1. Given the plane $x+2y-2z=8$ and the line $x=t,\quad y=1-t,\quad z=3+2t$;

a) Find the angle between the plane and the line. b) Find the point of intersection.



$x=t$, $y=1-t$, $z=3+2t$ $\Rightarrow \vec{d}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\vec{n}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$

$\beta=(90°-\theta)$ OR $\left(\frac{\pi}{2}-\theta\right)$

$$\cos\beta=\frac{\vec{d}\cdot\vec{n}}{|\vec{d}||\vec{n}|}=\frac{(i-j+2k)\cdot(i+2j-2k)}{\sqrt{1+1+4}\,\sqrt{1+4+4}}$$

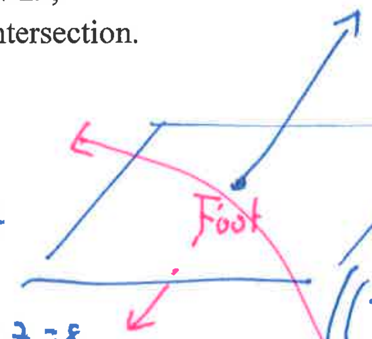
$$=\frac{1-2-4}{\sqrt{6}\,\sqrt{9}}=\frac{-5}{3\sqrt{6}}\Rightarrow \beta=\cos^{-1}\left(\frac{-5}{3\sqrt{6}}\right)=132.9°$$

$\beta_1=180-132.9=47.1°$

$\theta=90-47.1=42.9°$

l: $x=t$, $y=1-t$, $z=3+2t$

$x+2y-2z=8$



$(t)+2(1-t)-2(3+2t)=8$

$t+2-2t-6-4t=8$

$-5t-4=8$

$-5t=12$

$t=-\frac{12}{5}$

$x=\frac{-12}{5}$

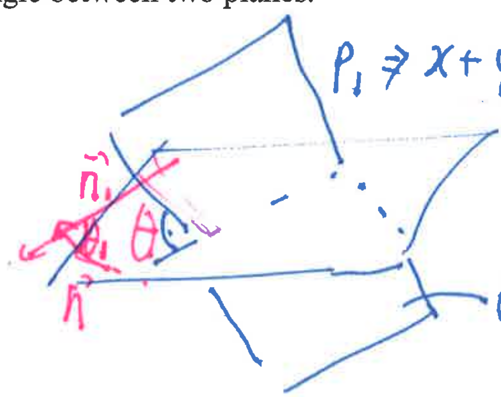
$y=1-\left(-\frac{12}{5}\right)=\frac{17}{5}$

$z=3+2\left(-\frac{12}{5}\right)=\frac{-9}{5}$

$\left(\frac{-12}{5},\frac{17}{5},\frac{-9}{5}\right)$

2. Given the planes $x+y-z=8$ and $2x-y+3z=-1$;

a) Find the angle between two planes.



$P_1 \Rightarrow x+y-z=8$

$\vec{n}_1=i+j-k$

$P_2=2x-y+3z=-1$

$\vec{n}_2=2i-j+3k$

$\theta_1=\theta$

$$\cos\theta_1=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}=\frac{(i+j-k)\cdot(2i-j+3k)}{\sqrt{1+1+1}\,\sqrt{4+1+9}}$$

$$=\frac{2-1-3}{\sqrt{3}\,\sqrt{14}}=\frac{-2}{\sqrt{3}\sqrt{14}}$$

$$\theta_1=\theta=\cos^{-1}\left(\frac{-2}{\sqrt{3}\sqrt{14}}\right)\approx\boxed{108°}$$

IB Math HL2 The Distance from a Point to a Plane

1. Find the parametric equations of the line through A(-1, 2, 3) and B(2, 0, -3). Hence find where this line meets the plane with equation $x - 2y + 3z = 26$.

$$\overleftrightarrow{AB} \Rightarrow l_1 = \begin{pmatrix}-1\\2\\3\end{pmatrix} + t\begin{pmatrix}2+1\\0-2\\-3-3\end{pmatrix} = \begin{pmatrix}-1\\2\\3\end{pmatrix} + t\begin{pmatrix}3\\-2\\-6\end{pmatrix} \Rightarrow \begin{aligned} x &= -1+3t\\ y &= 2-2t\\ z &= 3-6t.\end{aligned}$$

$x - 2y + 3z = 26$

$$\Rightarrow (-1+3t) - 2(2-2t) + 3(3-6t) = 26. \Rightarrow t = -2.$$

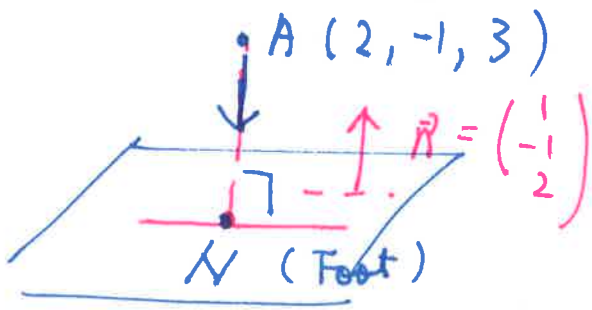
$x = -1 + 3(-2) = -7$

$y = 2 - 2(-2) = 6$

$z = 3 - 6(-2) = 15$

$(-7, 6, 15)$

2. Suppose N is the foot of the normal from A(2, -1, 3) to the plane $x - y + 2z = 27$. Find the coordinates of N, and hence find the shortest distance from A to the plane.



$$\overleftrightarrow{AN} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$x = 2 + t$

$y = -1 - t$

$z = 3 + 2t$

$$(2+t) - (-1-t) + 2(3+2t) = 27.$$

$t = 3$

$x = 2 + 3 = 5$

$y = -1 - (3) = -4$

$z = 3 + 2(3) = 9$

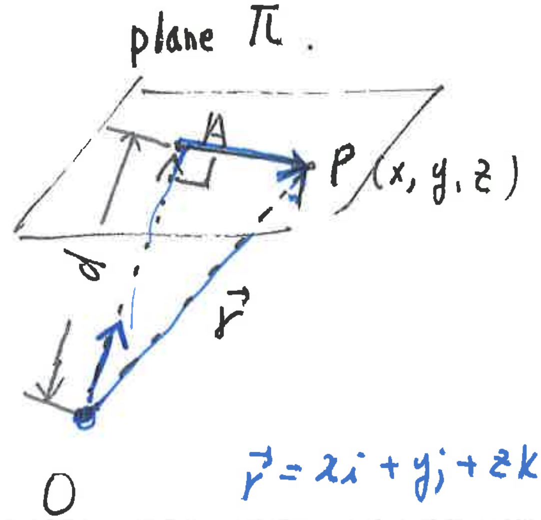
$N: (5, -4, 9)$ A (2, -1, 3)

$$d = \sqrt{(5-2)^2 + (-4+1)^2 + (9-3)^2} = \sqrt{54} = 3\sqrt{6} \text{ units}$$

$9 + 9 + 36$

- Distance from O to a plane.

$$\left(\vec{r} \cdot \hat{n} = d \right)$$



- $\hat{n}$ is an unit normal vector to the plane.
- $|\vec{OA}| = d$ (Distance O to plane)
- $\vec{OA} = d \cdot \hat{n}$

$\vec{r} = xi + yj + zk$.

- $\vec{AP} = \vec{AO} + \vec{r}$
- $\vec{AP} \perp \vec{OA}$
- $\vec{OA} = d \cdot \hat{n} \Leftrightarrow \vec{AO} = -d \cdot \hat{n}$

$$\vec{AP} \cdot \vec{AO} = 0$$

$$(\vec{AO} + \vec{r}) \cdot \vec{AO} = 0$$

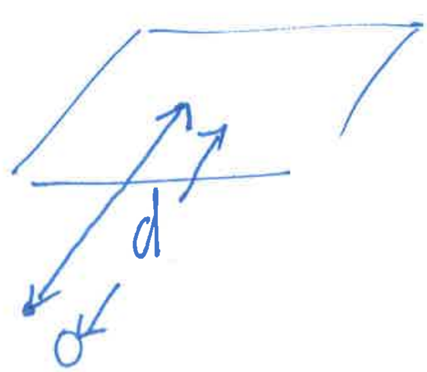
$$\frac{(-d\hat{n} + \vec{r}) \cdot (d \cdot \hat{n})}{d} = \frac{0}{d}$$

$$\Rightarrow (-d\hat{n} + \vec{r}) \cdot \hat{n} = 0.$$

$$\Rightarrow -d\,\hat{n} \cdot \hat{n} + \vec{r} \cdot \hat{n} = 0 \qquad (\hat{n} \cdot \hat{n} = 1)$$

$$\Rightarrow -d + \vec{r} \cdot \hat{n} = 0$$

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \Rightarrow \boxed{\vec{r} \cdot \hat{n} = d}$$

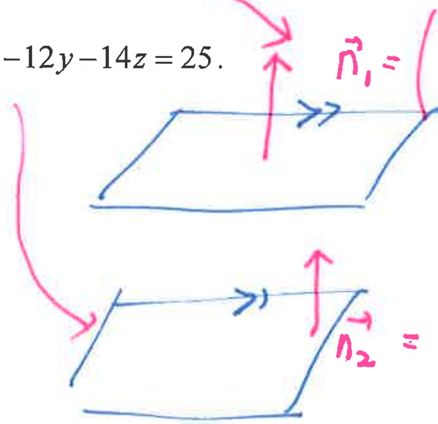


IB Math HL2 Distance line/line, line/plane, and plane/plane Name ______________

Distance:

EX) Given P1: $-3x+6y+7z=1$ and P2: $6x-12y-14z=25$.

a) Show above two planes are parallel



$$\vec{n}_1 = \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} \qquad \vec{n}_2 = \begin{pmatrix} 6 \\ -12 \\ -14 \end{pmatrix}$$

$$\Rightarrow k \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \\ -14 \end{pmatrix}$$

$k=-2$

$\therefore P_1 \parallel P_2$.

b) Find the shortest distance between two planes.

$$P_1 \Rightarrow \vec{r} \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} = 1$$

$$\vec{r} \cdot \hat{n} = d.$$

$$|\vec{n}_1| = \sqrt{9+36+49} = \sqrt{94}.$$

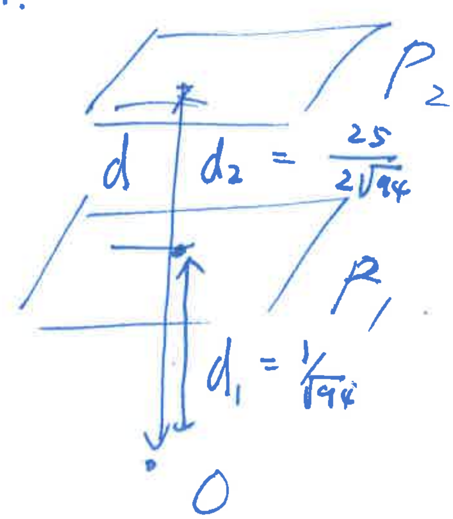
$$P_1: \vec{r} \begin{pmatrix} -3/\sqrt{94} \\ 6/\sqrt{94} \\ 7/\sqrt{94} \end{pmatrix} = 1/\sqrt{94}$$

$$P_2: \vec{r} \begin{pmatrix} 6 \\ -12 \\ -14 \end{pmatrix} = 25.$$

$$|\vec{n}_2| = \sqrt{36+144+196} = \sqrt{376}$$

$$\vec{r} \begin{pmatrix} 6/\sqrt{376} \\ -12/\sqrt{376} \\ -14/\sqrt{376} \end{pmatrix} = 25/\sqrt{374} = \frac{25}{2\sqrt{94}}.$$

$$d = |d_2 - d_1| = \left| \frac{25}{2\sqrt{94}} - \frac{1}{\sqrt{94}} \right|$$



EX) a) Find the equation of the plane determined by A (1, -2, 1), B(1, 0, -2) and C (-2, 6, -6).

$\vec{AB} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$

$\Rightarrow \vec{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 8 \\ -7 \end{pmatrix}$: vector Equation

$\vec{AC} = \begin{pmatrix} -2-1 \\ 6+2 \\ -6-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -7 \end{pmatrix}$

$\begin{vmatrix} i & j & k \\ 0 & 2 & -3 \\ -3 & 8 & -7 \end{vmatrix} = \begin{matrix} i\,(-14+24) \\ -j\,(-9) \\ k\,(6) \end{matrix} = 10i + 9j + 6k.$

$\Rightarrow 10x + 9y + 6z = D$

$10 - 18 + 6 = -2$

$\Rightarrow 10x + 9y + 6z = -2$ Cartesian Equation.

b) Show the plane from part a is parallel to the line $x - 3 = y + 3 = -6z - 48$. **Hence**, find the shortes distance from the line to the plane.

line: $\dfrac{x-3}{1} = \dfrac{y+3}{1} = \dfrac{z+8}{-6}$

Plane: $10x + 9y + 6z = -2$

$\Rightarrow$ If the line and plane are //, then the normal vector to the plane and the directional vector should be $\perp$.

$\begin{pmatrix} 10 \\ 9 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \Rightarrow 10 + 9 - 36 \neq 0.$

$\therefore$ The line and the plane are Not parallel.

$\Rightarrow$ Hence the distance is '0'.