

Partial Fraction Decomposition

Notes:

Objective: Learn how to integrate rational functions by using Partial fraction decomposition skill.

Read Calculus by Luson p555 for additional explanation.

1. Linear Factors	Ex) $\frac{-x+1}{x^2+3x+2} = \frac{2}{x+1} + \frac{-3}{x+2}$
2. A Repeated Linear Factor	Ex) $\frac{x^2-6x+3}{(x-2)^3} = \frac{1}{x-2} + \frac{-3}{(x-2)^2} + \frac{-5}{(x-2)^3}$ Ex) $\frac{5x^2+21x+4}{(x+1)^2(x-3)} = \frac{3}{(x+1)^2} + \frac{-2}{x+1} + \frac{7}{x-3}$
3. A Repeated Quadratic Factor	Ex) $\frac{-3x^3-x}{(x^2+1)^2} = \frac{-3x}{x^2+1} + \frac{2x}{(x^2+1)^2}$
4. Distinct Linear and Quadratic factors	Ex) $\frac{x^2+4x-23}{(x^2+4)(x+3)} = \frac{3x-5}{x^2+4} + \frac{-2}{x+3}$

In factored form:
 ◎ If denominator is linear, the numerator is constant.
 ◎ If denominator is Quadratic, the numerator is linear.

1. $\frac{-x+1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$ (Find A and B: The answer is in above #1)

$\frac{A(x+2) + B(x+1)}{(x+1)(x+2)} = \frac{-x+1}{(x+1)(x+2)}$ where $x \neq -1$ and $x \neq -2$

$A(x+2) + B(x+1) = -x+1$

⇒ Substitute $x = -1$ ⇒ $A = 2$

⇒ Substitute $x = -2$ ⇒ $B = -3$

2. $\frac{x^2-6x+3}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$

∴ $\frac{2}{x+1} + \frac{-3}{x+2}$

$\frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3} = \frac{x^2-6x+3}{(x-2)^3}$ where $x \neq 2$.

$A(x-2)^2 + B(x-2) + C = x^2-6x+3$

⇒ Substitute $x = 2$ $C = (2)^2 - 6(2) + 3 = -5$

∴ $\frac{1}{(x-2)} + \frac{-3}{(x-2)^2} + \frac{-5}{(x-2)^3}$

⇒ Substitute $x = 1$ $A - B - 5 = 1^2 - 6 + 3$

① $A - B = 3$ ← Solve the system of Equations

⇒ Substitute $x = 0$ $4A - B - 5 = 3$ ⇒ ② $4A - B = 8$ $A = 1$ $B = -2$

$$3. \frac{5x^2 + 21x + 4}{(x+1)^2(x-3)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-3}$$

$$\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-3} = \frac{5x^2 + 21x + 4}{(x+1)^2(x-3)} \quad \text{where } x \neq -1 \text{ and } x \neq 3.$$

$$A(x-3) + B(x+1)(x-3) + C(x+1)^2 = 5x^2 + 21x + 4$$

$$\bullet x=3 \Rightarrow C(4)^2 = 2(3)^2 + 21(3) + 4 \Rightarrow C = 7$$

$$\bullet x=-1 \Rightarrow A(-4) = 5(1)^2 + 21(1) + 4 \Rightarrow A = 3$$

$$\bullet x=0 \Rightarrow (3)(-3) + (B)(1)(-3) + C(1)^2 = 4 \Rightarrow B = -2$$

$$4. \frac{-3x^3 - x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{-3x^3 - x}{(x^2+1)^2}$$

$$(Ax+B)(x^2+1) + (Cx+D) = -3x^3 - x$$

$$\bullet x=0 \Rightarrow B(1) + D = 0 \Rightarrow \boxed{B=0}$$

$$\bullet x=1 \Rightarrow 2A + D = -2$$

$$\bullet x=-1 \Rightarrow 2A - D = 2$$

Solve for the system $\Rightarrow \begin{cases} A = -3 \\ D = 2 \end{cases}$

$$\therefore \frac{3}{(x+1)^2} + \frac{-2}{x+1} + \frac{7}{x-3}$$

$$\therefore \frac{-3x}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

$$5. \frac{x^2 + 4x - 23}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3}$$

$$\frac{Ax+B}{x^2+4} + \frac{C}{x+3} = \frac{x^2 + 4x - 23}{(x^2+4)(x+3)}$$

$$(Ax+B)(x+3) + C(x^2+4) = x^2 + 4x - 23 \quad (\text{where } x \neq -3)$$

$$\bullet x=-3 \Rightarrow (C)(9+4) = (-3)^2 + 4(-3) - 23 \Rightarrow C = -2$$

$$\bullet x=0 \Rightarrow B(3) + (-2)(4) = -23 \Rightarrow B = -5$$

$$\bullet x=1 \Rightarrow (A-5)(3) + (-2)(5) = 1 + 4 - 23 \Rightarrow A = 3$$

$$\therefore \frac{3x-5}{x^2+4} + \frac{-2}{x+3}$$

Integrate:

1. $\int \frac{-x+1}{x^2+3x+2} dx$

2. $\int \frac{x^2-6x+3}{(x-2)^3} dx$

3. $\int \frac{5x^2+21x+4}{(x+1)^2(x-3)} dx$

4. $\int \frac{-3x^3-x}{(x^2+1)^2} dx$

5. $\int \frac{x^2+4x-23}{(x^2+4)(x+3)} dx$

$$\frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x-3} = \frac{5x^2+21x+4}{(x+1)^2(x-3)}$$

$$\frac{A(x+1)(x-3) + B(x-3) + C(x+1)^2}{(x+1)^2(x-3)} = \frac{5x^2+21x+4}{(x+1)^2(x-3)}$$

$x \neq -1 \quad x \neq 3$

$$\Rightarrow A(x+1)(x-3) + B(x-3) + C(x+1)^2 = 5x^2+21x+4$$

• $x=3 \Rightarrow 4^2 = 5(4)^2 + 21(4) + 4 \Rightarrow C=7$

• $x=-1 \Rightarrow B(-4) = 5(-1)^2 + 21(-1) + 4 \Rightarrow B=3$

• $x=0 \Rightarrow A(1)(-3) + 3(-3) + 7(1)^2 = 4 \Rightarrow A=-2$

$$\begin{aligned} & \int \left(\frac{-2}{x+1} \right) dx + \int \frac{3}{(x+1)^2} dx + \int \frac{7}{x-3} dx \\ &= -2 \ln|x+1| - \frac{3}{x+1} + 7 \ln|x-3| + C \\ &= \ln \left(\frac{(x-3)^7}{(x+1)^2} \right) - \frac{3}{x+1} + C \end{aligned}$$

Practice) Find each integral by method of partial fraction.

a. $\int \frac{x+4}{(x-1)(x+6)} dx$

b. $\int \frac{1}{x^3+x^2-2x} dx$

c. $\int \frac{2x+1}{x^2-7x+12} dx$

d. $\int \frac{2x-1}{(x-1)^2} dx$

e. $\int \frac{1}{(x+1)(x^2+1)} dx$

f. Challenge $\int \frac{x^2+3x+1}{x^3-1} dx$