14K The Vector Product of Two Vectors (aka Cross Product)

The **vector cross product** of vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is $\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$

Tip: Use the determinant! $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Remember the signs! $+, -, +$

Examples: Use $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ to evaluate the expressions.

1. $\mathbf{a}\times\mathbf{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i}(1\cdot 3 - (-1)(2)) - \hat{j}(6+1) + \hat{k}(4-1)$$
$$= \boxed{5\hat{i} - 7\hat{j} + 3\hat{k}}$$

2. $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{a}$

$$\begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 10 - 7 - 3 = \boxed{0}$$

3. $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b}$

$$\begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5 - 14 + 9 = \boxed{0}$$

4. $\mathbf{a}\times\mathbf{a} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{matrix} \hat{i}(-1+1) \\ -\hat{j}(-2+2) \\ \hat{k}(2-2) \end{matrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

5. $-\mathbf{b}\times\mathbf{a}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \begin{matrix} \hat{i}(2+3) \\ -\hat{j}(1+6) \\ \hat{k}(-1+4) \end{matrix} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

6. $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 3 & 2 & 7 \end{vmatrix} = \begin{matrix} \hat{i}(7+2) \\ -\hat{j}(14+3) \\ \hat{k}(4-3) \end{matrix} = \begin{pmatrix} 9 \\ -17 \\ 1 \end{pmatrix}$$

7. $\mathbf{a}\times\mathbf{b}$

$$\begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix}$$

8. $\mathbf{a}\times\mathbf{c}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 0 & 4 \end{vmatrix} = \begin{matrix} \hat{i}(4-0) \\ -\hat{j}(8+2) \\ \hat{k}(0-2) \end{matrix}$$
$$= \begin{pmatrix} 4 \\ -10 \\ -2 \end{pmatrix}$$

Algebraic Properties of the Vector Cross Product Use the results of problems 1 – 8 to fill in the blanks.

$\mathbf{a}\times\mathbf{b}$ is a vector which is perpendicular to both **a** and **b**. (#1 – 3)

$\mathbf{a}\times\mathbf{a} = \vec{0}$ for all **a**.

$\vec{a}\times\vec{b} = -\mathbf{b}\times\mathbf{a}$ for all **a** and **b**.

Hence $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ have the same length and opposite direction.

$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$ is called the **scalar triple product**.

$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ (#6 – 8)

$(\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d}) = (\mathbf{a}\times\vec{c}) + (\mathbf{a}\times\vec{d}) + (\mathbf{b}\times\vec{c}) + (\mathbf{b}\times\vec{d})$

14K.1 (1ac, 2-6, 9)