IB Questions Solution (Differential Equations)

#1. $x\frac{dy}{dx} - y^2 = 1$

$x\frac{dy}{dx} = y^2 + 1$

$\frac{x}{dx} = \frac{y^2+1}{dy}$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y^2+1}$

$\ln x + c = \arctan y$. $(y=0, x=2)$

$\ln 2 + c = \arctan 0$

$c = -\ln 2$.

$\Rightarrow \arctan y = \ln x - \ln 2$.

$\arctan y = \ln \frac{x}{2}$

$y = \tan\left(\ln \frac{x}{2}\right)$

#2. $v(s) = \arctan(\sin s)$

(a)

$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt}$.

$= \frac{dv}{ds} \cdot v$.

$= \frac{d \arctan(\sin s)}{ds} \cdot v$

$= \frac{1}{1 + (\sin s)^2} \cdot \cos(s) \cdot \arctan \sin(s)$

$a(s) = \frac{\cos(s) \cdot \arctan(\sin(s))}{1 + (\sin(s))^2}$.

(b) $0.25 = \frac{\cos(s) \cdot \arctan[\sin s]}{1 + \sin^2 s}$.

Solve by nSolve. GDC. $s \approx -2.97$

#3. $a = \frac{2s}{s^2+1}$

(a) $a(t) = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v$.

$\Rightarrow \frac{2s}{s^2+1} = \frac{dv}{ds} \cdot v$.

$\Rightarrow \int \frac{2s}{s^2+1} \cdot ds = \int v \cdot dv$

$(u = s^2+1 \quad du = 2s \cdot ds)$

$\ln|s^2+1| = \frac{1}{2}v^2 + c$

$s=1$
$v=2$.

$\ln 2 = \frac{1}{2} \cdot 2^2 + c$

$c = \ln 2 - 2$. $\longrightarrow$

$$\frac{1}{2}v^2 = \ln(s^2+1) + 2 - \ln 2.$$

$$\frac{1}{2}v^2 = \ln\left(\frac{s^2+1}{2}\right) + 2.$$

$$v^2 = 2\ln\left(\frac{s^2+1}{2}\right) + 4$$

$$v = \overset{+}{\cancel{-}}\sqrt{2\ln\left(\frac{s^2+1}{2}\right) + 4} \qquad v \geq 0.$$

$\boxed{\text{Given } s=1 \quad v=2}$.

(b) $v = \sqrt{2\ln\left(\overset{13}{\cancel{\tfrac{26}{2}}}\right) + 4}$

$$v = \sqrt{2\ln 13 + 4} \ \frac{m}{s}$$

#4.

(a) (i) $v(t) = -10t$

(ii) $v(10) = -100 \ \frac{m}{s}.$

(iii) $S(100) \Rightarrow S(t) = \int -10t \, dt = -5t^2 + C.$ $\qquad C = 1000$

$$S(t) = -5t^2 + 1000$$

$$S(10) = -5(10)^2 + 1000 = 500$$

(b) $a(t) = -10 - 5v \quad t \geq 10.$

$$\Rightarrow \frac{dt}{dv} = \frac{1}{\frac{dv}{dt}} = \frac{1}{a(t)} = \frac{1}{-10-5v}.$$

$$\int dt = \int \left(\frac{1}{-10-5v} \right) dv$$

$$t = -\frac{1}{5} \ln \left| -10-5v \right| + C. \Leftarrow \quad v(10) = -100 \; (t=10, v=-100)$$

$$10 = -\frac{1}{5} \ln \left| -10-5(-100) \right| + C$$

$$C = 10 + \frac{1}{5} \ln(490).$$

(c) $$t = \frac{1}{5}\ln(490) - \frac{1}{5}\ln\left|-10-5v\right| + 10$$

$$= \frac{1}{5}\ln\left[\frac{490}{-10-5v}\right] + 10.$$

$$\boxed{t = 10 + \frac{1}{5}\ln\left[\frac{98}{-2-v}\right]}$$

(d) $$\frac{1}{5}\ln\left[\frac{98}{-2-v}\right] = t - 10$$

$$\Rightarrow \ln\left[\frac{98}{-2-v}\right] = 5(t-10)$$

$$\Rightarrow \frac{98}{-2-v} = e^{5(t-10)}$$

$$\Rightarrow \frac{-98}{e^{5(t-10)}} = 2 + v \Rightarrow \boxed{v = \frac{-98}{e^{5(t-10)}} - 2} \text{ OR } \boxed{v = -98e^{5(10-t)} - 2}$$

(e) $S = \int v\,dt$

$= \int \left(-98\, e^{5(10-t)} - 2\right) dt$

$= \frac{-98}{-5} e^{5(10-t)} - 2t + C \quad \Leftarrow \begin{pmatrix} t=10 \\ S=500 \end{pmatrix}$ from (a)

$\hookrightarrow 500 = \frac{98}{5} \cdot e^{0} - (2)(10) + C$

$C = 500 + 20 - \frac{98}{5}.$

$\boxed{C = 500.4}$

$\boxed{S(t) = \frac{98}{5} e^{5(10-t)} - 2t + 500.4}$

(f) $0 = \frac{98}{5} e^{5(10-t)} - 2t + 500.4 \quad \Leftarrow$ GDC

n-solve. $t = ?$ use n-solve.

$\boxed{250 \text{ sec}}$

$y = \frac{98}{5} e^{50-5t} - 2t + 500.4$