key ①

Mathematical Induction: Proof of a conjecture.

Day one: Sequence and Series

Math Induction proof process for a given conjecture (statement):

Step 1: Show that the statement is true for an initial case, $n=1$.

Step 2: Assume that the statement is true for $n=k$ where $k \in Z^+$.

Step 3: Prove that the statement is true for $n=k+1$.

Step 4: $\therefore$ The statement is true for $n \in Z^+$

Tune of "Blowin' in the Wind" by Bob Dylan

Math Induction

How can you tell that a statement is true
For every value of n?
Well there's just no way you can try them all.
Why you could barely begin!
Is there a tool that can help us resolve
This infinite quand'ry we're in?

The answer, my friend, is knowin' induction.
The answer is knowin' induction!

First, you must find an initial case
For which the statement is true,
Then you must assume it's true for k
Then k+1 must work too!
Then all statements fall like dominos do.
Tell me how did we score such a coup?

The answer, my friend, is knowin' induction.
The answer is knowin' induction!

Written by Arthur Benjamin

Problem 1)

Prove using Mathematical induction that $P(n)$ is true for all $n \geq 1$.

$$P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1) when $n=1 \Rightarrow 1 = \frac{(1)(1+1)}{2} = 1 \Rightarrow p(n)$ is true for $n=1$

2) Assume $p(n)$ is true for $n=k$ where $k \in Z^+ \Rightarrow \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

QED

3) If $n=k+1$, $\boxed{\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}}$

$\Rightarrow \sum_{i=1}^{k} i + k+1 = \frac{k(k+1)}{2} + k+1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \boxed{\frac{(k+1)(k+2)}{2}}$.

4) $\therefore$ $p(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ is true for all $n \geq 1$.

Problem 2)

Prove the equality holds for all $n \geq 1$.

$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$ ⇒ E: Equality

1) when $n=1$ ⇒ $1 \cdot 1! = (1+1)! - 1 = 2 - 1 = 1$ ⇒ $1 = 1$: The Equality is true for $n=1$

2) when $n = k$ where $k \in \mathbb{Z}^+$, Assume the Equality is true.

$$\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$$

3) When $n = k+1$: $\boxed{\sum_{i=1}^{k+1} = (k+2)! - 1}$ QED

$$\sum_{i=1}^{k} i \cdot i! + [(k+1)(k+1)!] = (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k)(k+1)! + (k+1)!$$

$$= 2(k+1)! + k(k+1)! - 1 = (k+1)!(k+2) - 1 = \boxed{(k+2)! - 1}$$

4) ∴ The Equality $\sum_{i=1}^{n} i \, i! = (n+1)! - 1$ is true for $n \geq 1$.

IB Questions) Do on separate paper

1 Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

2 a) Use mathematical induction to prove that

$$P(n): \sum_{i=1}^{n} i(i+4) = \frac{n(n+1)(2n+1)}{6}$$

b) Hence find $\sum_{i=40}^{60} i(i+4)$

Day one IB Questions solution.

#1. $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 \cdots n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$: $p(n)$

1) For $n=1$

$$1 = 4 - \frac{1+2}{1} = 4 - 3 = 1 \Rightarrow p(n) \text{ is true}$$

2) Assume $p(n)$ is true for $n=k$

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 \cdots k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$$

3) If $n = k+1$

$$\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 \cdots k\left(\frac{1}{2}\right)^{k-1}\right] + (k+1)\left(\frac{1}{2}\right)^{k+1-1}$$

$$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$$

$$= 4 - \frac{2(k+2)}{2(2^{k-1})} + \frac{k+1}{2^k}$$

$$= 4 + \frac{-2k-4}{2^k} + \frac{k+1}{2^k}$$

$$= 4 + \frac{-k-3}{2^k}$$

$$= \boxed{4 - \left[\frac{k+3}{2^k}\right]}$$

QED.

$$= \boxed{4 - \left[\frac{k+3}{2^k}\right]}$$

$\therefore$ 4) $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 \cdots n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}$

is true for $n \in \mathbb{Z}^+$

p(n):

#2. $\sum_{i=1}^{n} i(i+4) = \frac{n(n+1)(2n+13)}{6}$

1) when $n=1$

$1(1+4) = \frac{(1)(2)(2+13)}{6} \to 5 = 5$, p(n) is true for $n=1$.

2) when $n=k$ ($k \in \mathbb{Z}^+$), Assume the statement is true.

$$\sum_{i=1}^{k} i(i+4) = \frac{k(k+1)(2k+13)}{6}$$

3) If $n=k+1$

$$\sum_{i=1}^{k+1} i(i+4) = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k} i(i+4) + (k+1)(k+1+4)$$

$$= \frac{k(k+1)(2k+13)}{6} + \frac{(k+1)(k+5)\cdot 6}{6}$$

$$= (k+1)\frac{[2k^2+13k+6k+30]}{6} = \frac{(k+1)(2k^2+19k+30)}{6}$$

$$= \frac{(k+1)(2k+15)(k+2)}{6} = \frac{(k+1)(k+2)[2(k+1)+13]}{6}$$

$2k^2 + 19k + 30$

15

2

$2k$

$2k$

k

QED