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GROUP #5: CURVE ANALYSIS

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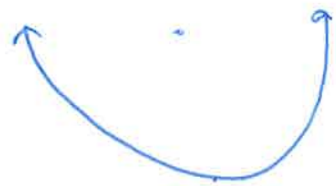
First Derivative Test:

$(c, f(c))$ is a local max where $f'(c)=0$ and:

$f'(c)=0$

$f'(x)$ ⊕ c ⊖

$(c, f(c))$ is a local min where $f'(c)=0$ and:



$f'(c)=0$

$f'(x)$ ⊖ c ⊕

Neither min or max when:

$f'(x)$ ⊕ ⊕ or ⊖ ⊖

Second Derivatic Test:

If $f''(c)>0$ at c, $(c, f(c))$ is a relative min

CONCAVE UP

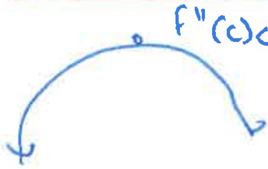
$f''(c)>0$

$f''(c)$ c ⊕

If $f''(c)<0$, at $x=c$, c is a relative max

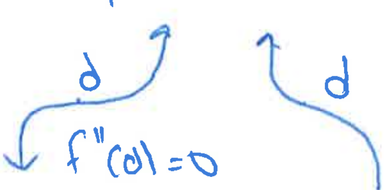
CONCAVE DOWN

$f''(c)<0$



$f''(c)$ c ⊖

If $f''(d)=0$, inflection point

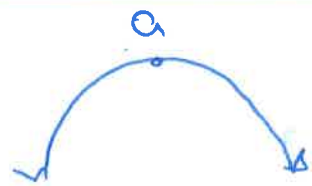
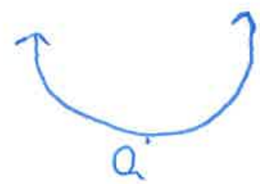
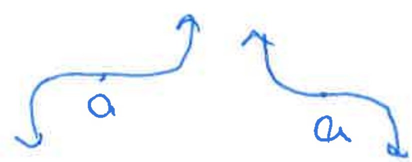


$f''(d)=0$

$f''(c)$ ⊖ d ⊕ / ⊕ d ⊖

② Increasing and decreasing !

- take first derivative of a function
- factor and draw sign diagram
- if +, increasing
- if −, decreasing

$f'(x)=0$	$f'(x)$ diagram	Graph
Local Max:	+ $\longrightarrow$ −	
Local Min:	− $\longrightarrow$ +	
inflection	+ $\rightarrow$ + or − $\rightarrow$ −	

1) consider $f(x) = x^3(x-2)$

a) Find the x coordinates of the stationary points of f.

$f(x) = x^4 - 2x^3$

$f'(x) = 4x^3 - 6x^2$

$\boxed{x = 0,\ 3/2}$

$0 = 2x^2(2x-3)$

$2x^2 = 0 \qquad 2x-3=0$

$x = 0 \qquad x = 3/2$

b) Determine intervals where the curve is increasing. Justify with a sign diagram

Sign Diagram of $f'(x)$

$\therefore$ increasing interval is $(3/2, \infty)$

c) Find $f''(x)$, and then determine the point(s) of inflection.

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x$

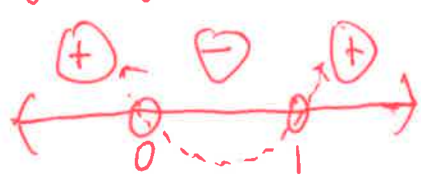
$0 = 12x(x-1)$

$12x = 0 \qquad x-1=0$

$x = 0 \qquad x = 1$

$$\boxed{\begin{gathered} f''(x) = 12x^2 - 12x \\ x = 0, 1 \end{gathered}}$$

Sign Diagram of $f''(x)$



d) Classify each point.

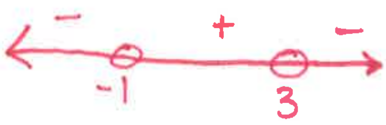
Local Max: n/a   Local min: $x = 3/2$ Stationary-inflection: $x = 0$

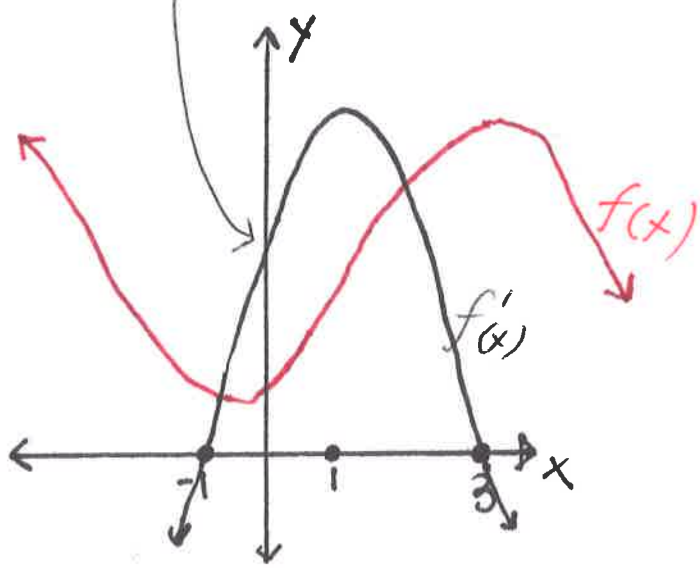
Nonstationary-inflection: $x = 1$

2) The diagram shows part of the graph of the gradient (slope) function, $y = f'(x)$ ④

a) On the same axes, sketch a graph of $y=f(x)$, clearly indicating the x values of local min and local max

Sign Diagram for $f'(x)$





b) Complete the table for the graph of $y=f(x)$

Inflection	x-values
i) inflection point on f	$x = 1$
ii) decreasing interval on f	$(-\infty, -1) \cup (3, \infty)$

3) Consider the function defined by $f(x) = x^3 - 3x^2 + 4$

a) Determine values of x for which $f(x)$ is a decreasing function

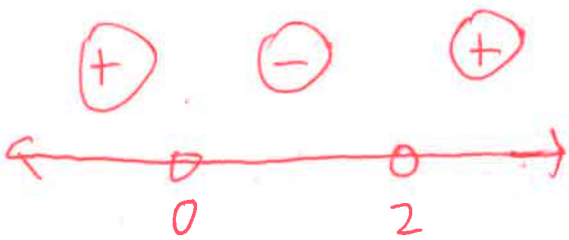
$$\frac{df}{dx} = 3x^2 - 6x = 0$$

$$= 3x(x-2) = 0$$

$3x = 0$ $\quad$ $x - 2 = 0$

$x = 0$ $\quad$ $x = 2$

Sign Diagram of $f'(x) = 3x(x-2)$



Decreasing Interval

$(0, 2)$

or $0 < x < 2$

b) There is a point of inflection P on the curve $y = f(x)$.
Find the coordinates of P

$$\frac{d^2f}{dx^2} = 6x - 6 = 0$$

$$= 6x = 6$$

$$x = 1$$

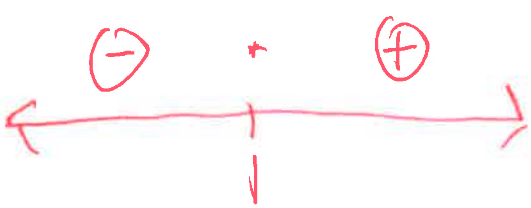
$f(x) = x^3 - 3x^2 + 4$

$f(1) = 1^3 - 3(1)^2 + 4$

$= 1 - 3 + 4$

$= 2$

Sign Diagram of $f''(x) = 6(x-1)$



$\boxed{P(1, 2)}$

Related Rates

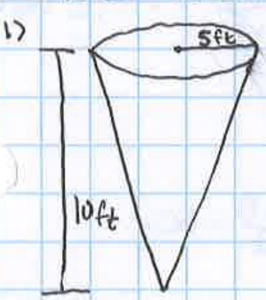
· Find the rate a quantity changes

5 Steps to solve
1. Draw a large, clear diagram
2. Write down information and determine variables & constants
3. Write Equation connecting the variables
4. Differentiate equation with respect to t
5. Solve for particular case

Example

1. Water runs into a conical tank standing point down at the rate of $9\,ft^3/min$. The tank's height is 10 feet and its base radius is 5 feet. How fast is the water level rising when the water is 6 feet deep?

1)

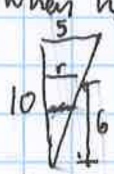


2) $\frac{dv}{dt} = 9$ $\quad \frac{dh}{dt}$ when $h=6$ is ?

3) $V = \frac{1}{3}\pi r^2 h$

4) $\frac{dv}{dt} = \frac{1}{3}\pi\, 2r \frac{dr}{dt} h + \frac{1}{3}\pi r^2 \frac{dh}{dt} \rightarrow \frac{dv}{dt} = \frac{\pi}{3}\left[2r\left(\frac{dr}{dt}\right)h + r^2\left(\frac{dh}{dt}\right)\right]$

When $h=6$



$\frac{5}{10} = \frac{r}{6}$ $\quad r=3$ $\qquad \frac{5}{10} = \frac{r}{h}$ $\qquad 5h = 10r$

$h = 2r$

taking derivitive

$\left(\frac{1}{2}\right)\frac{dh}{dt} = 2\frac{dr}{dt}\left(\frac{1}{2}\right.$

$\frac{1}{2}\frac{dh}{dt} = \frac{dr}{dt}$

$r=3$ $\quad h=6$

$\frac{1}{2}\frac{dh}{dt} = \frac{dr}{dt}$ $\quad \frac{dv}{dt} = 9$

$$\frac{dv}{dt} = \frac{\pi}{3}\left[2(r)\frac{dr}{dt}h + r^2\left(\frac{dh}{dt}\right)\right]$$

$$9 = \frac{\pi}{3}\left[\cancel{2}(3)\left(\frac{1}{\cancel{2}}\frac{dh}{dt}\right)(6) + 3^2\frac{dh}{dt}\right]$$

$$9 = \frac{\pi}{3}\left(18\frac{dh}{dt} + 9\frac{dh}{dt}\right)$$

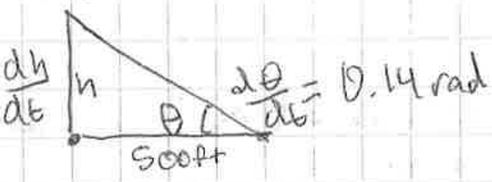
$$9 = \frac{\pi}{3}\left(27\frac{dh}{dt}\right)$$

$$\frac{9}{9\pi} = \frac{9\pi\frac{dh}{dt}}{9\pi}$$

$$\boxed{\frac{dh}{dt} = \frac{1}{\pi}\ ft/min}$$

2. A hot-air balloon rising straight up from a field is tracked by a range finder 500 feet from the lift-off point. At the moment the range-finder's angle of elevation is $\frac{\pi}{4}$, The angle is increasing at a rate of 0.14 rad/min. How fast is the balloon rising in this moment?

1)



2) $\tan\theta = \frac{h}{500}$

3) $\sec^2\theta \frac{d\theta}{dt} = \frac{1}{500}\left(\frac{dh}{dt}\right)$

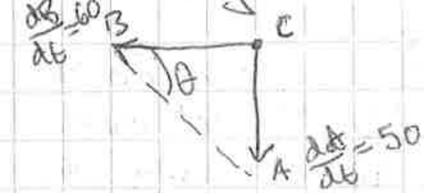
4) $\sec^2\left(\frac{\pi}{4}\right)(0.14\text{ rad}) = \frac{1}{500}\frac{dh}{dt}$

$500\left(\sec^2\frac{\pi}{4}(0.14)\right) = \frac{dh}{dt}$

$\frac{dh}{dt} = 140 \text{ ft/min}$

3. Car A and B leave town C at the same time. Car A heads South at a rate of 50 Km/hr, and car B heads west at 60 Km/hr. What rate is angle $C\hat{B}A$ changing after 3 hours?

1.



2. $\tan\theta = \frac{A}{B}$

3. $\sec^2\theta \frac{d\theta}{dt} = \frac{\frac{dA}{dt}(B) - \frac{dB}{dt}(A)}{B^2}$

T = 3hrs After 3 hrs A will go $\frac{50\text{Km}}{1\text{hr}} \cdot 3\text{hrs} = 150\text{Km}$

After 3 hrs B will go $\frac{60\text{Km}}{1\text{hr}} \cdot 3\text{hr} = 180\text{KM}$

$\tan\theta = \frac{150}{180}$

$\theta = \tan^{-1}\frac{150}{180}$

$\theta = 39.8^\circ$

4. $\sec^2\theta \frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - \frac{dB}{dt}A}{B^2}$

$\theta = 39.8$ $B = 180$ Km $A = 150$ Km $\frac{dA}{dt} = 50$ $\frac{dB}{dt} = 60$

$\sec^2 39.8 \frac{d\theta}{dt} = \frac{50 \cdot 180 - 150 \cdot 60}{180^2}$

$\frac{d\theta}{dt} = \frac{50 \cdot 180 - 150 \cdot 60}{180^2(\sec^2 39.8)}$

$\frac{d\theta}{dt} = 0/\text{hr}$

Related Rates Review:

$\frac{dy}{dx}$ = rate of change in y with respect to x

- take in account that it needs to be differentiated with respect to time

Example #1.

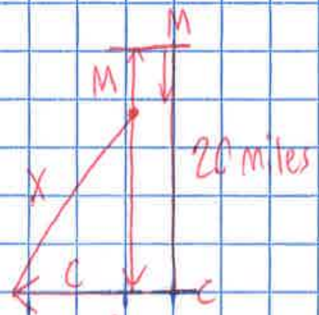
An underground conical tank, standing on its vertex is being filled with water at the rate of $48\ ft^3/min$. The tank has a height of 42 ft and radius of 30 ft. How fast is the water top surface

Optimization Review:

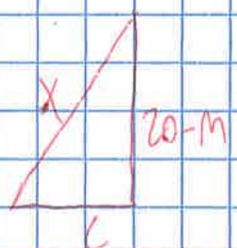
- look for largest or smallest values a function can take, along with a form of constraint → some condition that must always be followed
  - ex. area of rectangle HAS to be $24\ cm^2$, cannot change area
    - ↳ problem will ask to solve for largest perimeter possible, but area will always remain the same.

Related Rate Problem:

At 4:00 pm Bill is riding his horse 20 miles due south of Phil riding his horse on the same open plain. If Bill's horse is going 18 mph west and Phil's horse is going 22 mph south, at what rate is the distance between Bill and Phil changing at 5:00 pm on the same day?



$\frac{dc}{dt} = 18$ mph



after $t = 1$ hour

$M = (22)(1) = 22$ miles $\quad C = (18)(1) = 18$ mile

$X^2 = C^2 + (20 - M)$

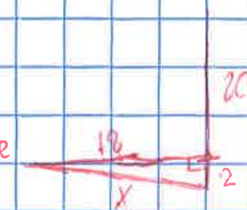
$= C^2 + (20)^2 - 40M + M^2$

$2X \boxed{\frac{dx}{dt}} = 2C \cdot \frac{dc}{dt} - 40 \frac{dm}{dt} + 2M \frac{dm}{dt}$

$2(\sqrt{328}) \cdot \frac{dx}{dt} = 2(18)(18) - 50(22) + 2(22)(22)$

$\frac{dx}{dt} = \frac{258}{2\sqrt{328}}$ mi/hr or $\frac{dx}{dt} = 14.2$ miles per hour

$\frac{dx}{dt} = ?$



$X = \sqrt{328}$ miles

Optimization

~ find the most efficient solutions

1. Draw a diagram
2. Construct a formula with the variable to be optimized
3. Find the 1st derivative and solve for x
4. Confirm if solution is a maximum or minimum and revisit if the solution is reasonable

Jennifer makes daily trips from the math building to the English building as shown. Her speed is 5ft/s on sidewalk and 4 ft/s on the grass. What is the least time her route could take.

1

2

$$T = \frac{\sqrt{500^2 + x^2}}{4} + \frac{800 - x}{5}$$

$$T = \frac{1}{4}(500^2 + x^2)^{\frac{1}{2}} + \frac{800}{5} - \frac{x}{5}$$

(3) $T = \frac{1}{4}(500^2 + x^2)^{\frac{1}{2}} + \frac{800}{5} - \frac{x}{5}$

(2)

$$\frac{dT}{dx} = \frac{1}{8}(500^2 + x^2)^{-\frac{1}{2}}(2x) - \frac{1}{5}$$

$$\cancel{\frac{dT}{dx}}\ 0 = \frac{2x}{8\sqrt{500^2 + x^2}} - \frac{1}{5}$$

$$\frac{1}{5} = \frac{2x}{8\sqrt{500^2 + x^2}}$$

$$8\sqrt{500^2 + x^2} = 10x$$

$$\sqrt{500^2 + x^2} = \frac{10}{8}x$$

$$500^2 + x^2 = \frac{25}{16}x^2$$

$$500^2 = \frac{9}{16}x^2$$

$$x = \sqrt{\frac{500^2}{9/16}}$$

$$x = 667$$

$$T = \frac{\sqrt{500^2 + (667)^2}}{4} + \frac{800 - 667}{5} = \boxed{235 \text{ seconds}}$$

$\frac{dT}{dx}$

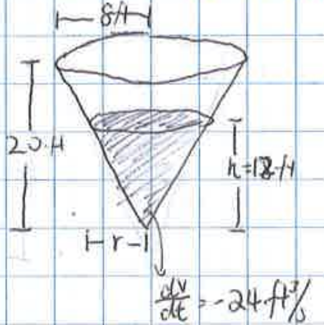
(4) $\ominus$ | $\oplus$

667

$\therefore$ minimum

Optimization and Related Rate Problems

1. A tank with water is in the shape of an inverted cone 20 ft high with a circular base on top whose radius is 8 ft. Water is running out the bottom at a rate of 24 ft^3/s. How fast is the water level falling when the water is 12 ft deep?



$$\frac{r}{h} = \frac{8}{20}$$

$$r = \frac{2}{5}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$$

$$V = \frac{4}{75}\pi h^3$$

$$\frac{dV}{dt} = \frac{4}{75}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{4}{25}\pi h^2 \cdot \frac{dh}{dt}$$

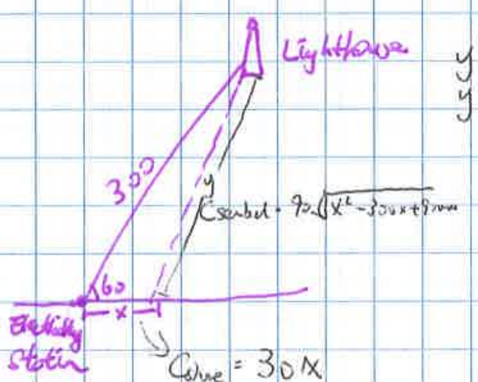
$$-24 = \frac{4}{25}\pi (12)^2 \cdot \frac{dh}{dt}$$

$$-\frac{25}{24\pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -0.332\ ft/s$$

The water level is falling 0.332 ft per second

2. A lighthouse is located 300 m from the electricity station. The angle between the coastline and the line joining the lighthouse is 60°. A cable needs to be laid between the station and lighthouse. If cable cost $~~[illegible]~~ 90/m on sea bed and $30/m and the distance laid on shore is x m, find x so the cost is minimized.



$$y^2 = x^2 + 300^2 - 2(x)(300)\cos 60$$

$$y^2 = x^2 - 300x + 90000$$

$$C = 30x + 90\sqrt{x^2-300x+90000}$$

$$\frac{dC}{dx} = 30 + \frac{90(2x-300)}{2\sqrt{x^2-300x+90000}} = 0$$

$$\frac{45\,(2x-300)}{\sqrt{x^2-300x+90000}} = -30$$

$$45(2x-300) = -30\sqrt{x^2-300x+90000}$$

$$6x - 900 = -2\sqrt{x^2-300x+90000}$$

$$(6x-900)^2 = 4(x^2-300x+90000)$$

$$36x^2 - 10800x + 810000 = 4x^2 - 1200x + 360000$$

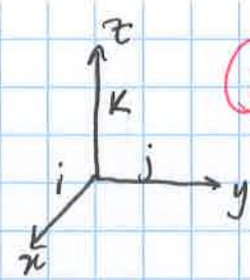
$$32x^2 - 9600x + 450000 = 0$$

$$x = 58.1,\ 241.8$$

$$\left.\frac{d^2C}{dx^2}\right|_{x=58.1} = 0.290 > 0$$

when $x = 58.1$ m cost is minimized

vectors



vector = a quantity w/ magnitude and direction

Notation: $\overrightarrow{PQ} = 3i + 4j + 5k = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

(on the same line)

- can be *parallell, equal, or zero, or *colinear
- *addition can be done w/ "tail to tip" method
- unit vectors have magnitude = 1

ex: $\overrightarrow{uv} = \dfrac{\overrightarrow{uv}}{|uv|}$

$a = t * b$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$\boxed{\text{Dot product}}$ → the scalar product

- if $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, the dot product is

$$\boxed{v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

- can be used in the equation $\cos\theta = \dfrac{v \cdot w}{|v||w|}$
- if $\theta = 90°$, $u \cdot w = 0$ (perpendicular!)

$\boxed{\text{cross product}}$ → the vector product

$$\boxed{a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}} = i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 b_2 - a_2 b_1)$$

- gives the perpendicular line

area of $\Delta = \frac{1}{2} |\vec{A} \times \vec{B}| = \frac{1}{2} |\vec{B}| \cdot |\vec{A}| \cdot \sin\theta$

Practice: find $\overrightarrow{BA}$ if B is (3, 2) and A is (2, 1)

$A - B = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \boxed{-i - j}$

if $\vec{n}$ and $\vec{B}$ are colinear (or parallell) and $a = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ k \end{pmatrix}$, find k

$a = t \cdot b$

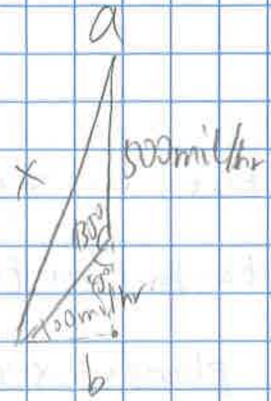
$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = t \cdot \begin{pmatrix} 4 \\ k \end{pmatrix}$ → $2 = t \cdot 4$, $t = \frac{1}{2}$ → $4 = \frac{1}{2}(k)$

$\boxed{k = 8}$

2

1. A airplane is flying from a to b. b is south of a. The plane's speed is 500 miles/hr. A wind is blowing constantly 100 miles from northeast.

a) find speed and direction of airplane

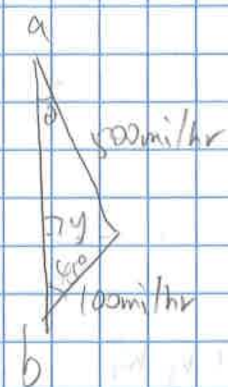


$$x^2 = 500^2 + 100^2 - 2 \cdot 500 \cdot 100 \cdot \cos 135°$$

$$x^2 = 250000 + 10000 + 70711$$

$$x = 575 \text{ mi/hr}$$

b) What direction must the airplane fly to compensate the wind



$$y = \frac{100}{\sqrt{2}} \text{ mi/hr}$$

$$\sin\theta = \frac{\frac{100}{\sqrt{2}}}{500}$$

$$\theta = 8.13° \text{ ES}$$

2. Consider $\vec{OA}$, a position vector from O to coordinate (3,5,2). Find the acute angle between $\vec{OA}$ and y-axis.

$$\begin{pmatrix}3\\5\\2\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\5\\0\end{pmatrix}$$

$$|\vec{OA}| = \sqrt{3^2+5^2+2^2} = \sqrt{38}$$

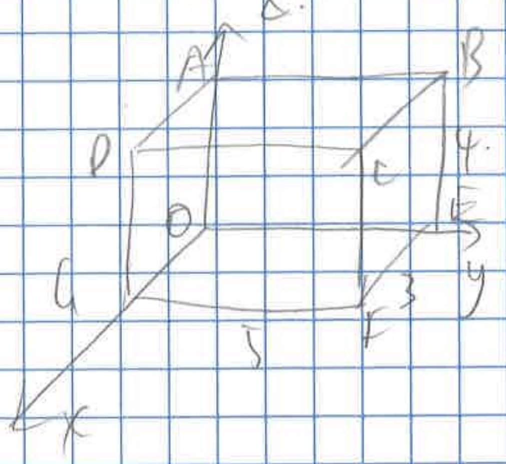
$$\cos\theta = \frac{5}{\sqrt{38}}$$

$$\theta = 35.8°$$

3

3.

Find $\vec{AE}$ and $\vec{AF}$!



a) $\vec{AE} = 0i + 5j - 4k$

b) $\vec{AF} = 3i + 5j - 4k$

c) Find the perpendicular vector of ΔAEF

$$\vec{AE} \times \vec{AF} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ -12 \\ -15 \end{pmatrix}$$

d) Find the area of ΔAEF

$$\text{area} = \frac{1}{2} \left| \vec{AE} \times \vec{AF} \right| = \frac{1}{2}\sqrt{12^2 + 15^2} = 9.6$$