CHAPTER 17 - limits

LIMITS

- Necessary for finding gradient of a tangent
- Solving: Simplify then substitute limit value into the function ($\lim_{x \to 0} \Rightarrow x = 0$)
- Continuous if: ① $f(a)$ is defined ② $\lim_{x \to a} f(x)$ exists ③ $f(a) = \lim_{x \to a} f(x)$

Ex. Evaluate $\lim_{x \to 2} \frac{\sqrt{x+3} - 2}{x - 1}$

Answer: $\lim_{x \to 2} \frac{\sqrt{x+3} - 2}{x-1} \frac{(\sqrt{x+3} + 2)}{(\sqrt{x+3} + 2)} \Rightarrow \lim_{x \to 2} \frac{x+3-4}{(x-1)(\sqrt{x+3} + 2)} \Rightarrow \lim_{x \to 2} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x+3} + 2)}$

$\frac{1}{\sqrt{2+3} + 2} = \boxed{\frac{1}{\sqrt{5} + 2}}$

LIMITS AT infinity

$x \to \infty$ means x as large as we like and positive

$x \to -\infty$ means x as large as we like and negative

★ For a rational function, limit approaching to ∞...

Find the HORIZONTAL / OBLIQUE asymptote

LIMIT APPROACHING 0

$\lim_{u \to 0} \frac{\sin u}{u} = 1$

Ex $\lim_{x \to 0} \frac{\sin 4x}{7x}$

Answer: $\lim_{x \to 0} \left(\frac{\sin 4x}{4x}\right)\left(\frac{4}{7}\right) = (1) \cdot \left(\frac{4}{7}\right) = \boxed{\frac{4}{7}}$

TRIG LIMITS

If θ is in radians $\Longrightarrow \lim_{\theta \to 0} = \frac{\sin \theta}{\theta} = 1$

Ex find $\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}$

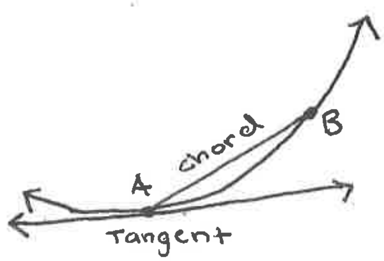
Answer: $= \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta} \times 3 = 3 \times \lim_{3\theta \to 0} \frac{\sin 3\theta}{\sin 3\theta} = 3 \cdot 1 = \boxed{3}$

RATES OF CHANGE

Rate = comparison between two quantities w/ different units

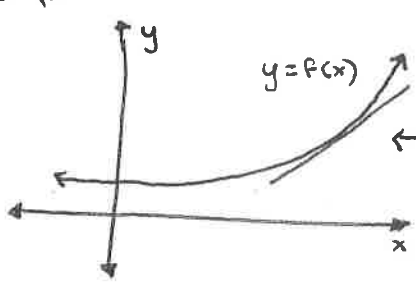
Slope = $\frac{\text{rise}}{\text{run}}$ average speed = $\frac{\text{distance traveled}}{\text{time taken}}$

instantaneous rate of change = gradient of tangent to graph at given point



DERIVATIVE FUNCTIONS

- gradient function of $y=f(x)$ is called its <u>derivative function</u> and is labelled $f'(x)$
- value of $f'(a)$ is the gradient of the tangent to $y=f(x)$ at the point where $x=a$



← equation of this tangent = gradient function = $y=f'(x)$

1st PRINCIPLE DIFFERENTIATION

The <u>derivative function</u> or simply <u>derivative</u> of $y=f(x)$ is...

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

<u>alternative notation</u>

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \Rightarrow f'(x)=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\boxed{\frac{dy}{dx}}$$

represents derivative when given y in terms

WHEN $x=a$

gradient of tangent to $y=f(x)$ at the point where $x=a$ is $f'(a)$ where...

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

1st PRINCIPLE derivative

Ex find the slope of the tangent line to the graph of $f(x) = -3x^2 + 2x$ at $x = 3$ using the 1st principle definition of derivatives: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Answer:

$$\lim_{h \to 0} \frac{-3(x+h)^2 + 2(x+h) - (-3x^2 + 2x)}{h}$$

$$\lim_{h \to 0} \frac{-3(3+h)^2 + 2(3+h) - (-3(3)^2 + 2(3))}{h}$$

$$\lim_{h \to 0} \frac{-3(h^2 + (h+9)) + 6 + 2h - (-27+6)}{h}$$

$$\lim_{h \to 0} \frac{-3h^2 - 18h \cancel{-27} \cancel{+6} + 2h \cancel{+27} \cancel{-6}}{h}$$

$$\lim_{h \to 0} \frac{-3h^2 - 18h + 2h}{h} = \lim_{h \to 0} \frac{-3h^2 - 16h}{h}$$

$$\lim_{h \to 0} \frac{h(-3h - 16)}{h}$$

$$\lim_{h \to 0} -3h - 16 = -3(0) - 16 = \boxed{-16}$$

CHAPTER 18:

Rules of Differentiation

$f(x)$	$f'(x)$
c (constant)	0
x^n	nx^{n-1}
$c\cdot u(x)$	$c\cdot u'(x)$
$u(x)+v(x)$	$u'(x)+v'(x)$

Derivatives of Exponential Functions:

If $f(x)=e^x$, $f'(x)=e^x$.

If $y=e^{f(x)}$, $\frac{dy}{dx}=f'(x)\cdot e^{f(x)}$

If $y=a^x$ where $a>0$, $\frac{dy}{dx}=a^x \ln a$.

Chain Rule:

If $y=g(u)$ and $u=f(x)$,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

↳ $y=[f(x)]^n$, $\frac{dy}{dx}=n[f(x)]^{n-1}\times f'(x)$

Derivatives of Log Functions

If $y=\ln x$, $\frac{dy}{dx}=\frac{1}{x}$

If $y=\ln(f(x))$, $\frac{dy}{dx}=\frac{f'(x)}{f(x)}$

Product Rule:

If $u(x)$ and $v(x)$ are two functions of x and $f(x)=u(x)\cdot v(x)$,

$$\frac{d}{dx}f(x)=u'(x)v(x)+u(x)v'(x)$$

Derivatives of Trigonometric Functions:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$

Quotient Rule:

If $Q(x)=\frac{u(x)}{v(x)}$,

$$Q'(x)=\frac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$$

Implicit Differentiation:

When it is difficult or impossible to write y as a function of x, we can still find $\frac{dy}{dx}$ from implicit relations.

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Problems:

#1: Find $\frac{dy}{dx}$ for $x^3e^y + y^2 \ln x = 20x$.

$$x^3e^y\frac{dy}{dx} + 3x^2e^y + y^2\cdot\frac{1}{x} + 2y\frac{dy}{dx}\ln x = 20$$

$$\frac{dy}{dx}\left(x^3e^y + 2y\ln x\right) = 20 - 3x^2e^y - \frac{y^2}{x}$$

$$\frac{dy}{dx} = \frac{20 - 3x^2e^y - \frac{y^2}{x}}{x^3e^y + 2y\ln x}$$

$$= \frac{20x - 3x^3e^y - y^2}{x^4e^y + 2xy\ln x}$$

#2: Find the derivative of $f(x) = \frac{12}{\sqrt[7]{x^2}} - \frac{x^7}{3}$

$$\frac{df}{dx} = 12\left(-\frac{2}{7}\right)x^{-\frac{2}{7}-\frac{7}{7}} - \frac{1}{3}(7)x^{7-1}$$

$$\frac{df}{dx} = \frac{-24}{7}x^{-9/7} - \frac{7}{3}x^6$$

$$\frac{df}{dx} = -\frac{24}{7x\sqrt[7]{x^2}} - \frac{7x^6}{3}$$

#3. Find the derivative of $f(x) = \frac{\tan 5x}{6x^4 - x^2}$.

$$\frac{df}{dx} = \frac{(6x^4-x^2)(\sec^2(5x))\cdot 5 - (\tan(5x))\cdot(24x^3-2x)}{(6x^4-x^2)^2} =$$

$$\frac{(30x^4-5x^2)(\sec^2(5x)) - (24x^3-2x)(\tan 5x)}{(x(6x^3-x))^2} =$$

$$\frac{\cancel{x}\left[(30x^3-5x)(\sec^2(5x)-(24x^2-2)(\tan 5x)\right]}{x^{\cancel{2}}(6x^3-x)^2} =$$

$$\frac{(30x^3-5x)\sec^2(5x)-(24x^2-2)(\tan 5x)}{x\cdot\left(x(6x^2-1)\right)^2} =$$

$$\frac{(30x^3-5x)\sec^2(5x)-(24x^2-2)(\tan 5x)}{x^3(6x^2-1)^2}$$

Chapter 19 (Group 3)

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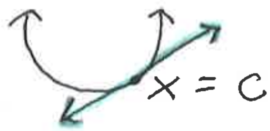
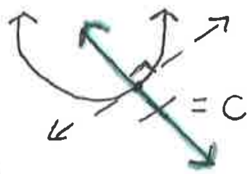
Finding the equation of a TANGENT line to the graph of f(x) at x = c

$$y - f(c) = f'(c)(x - c)$$



Finding the equation of a NORMAL line to the graph of f(x) at x = c

$$y - f(c) = -\frac{1}{f'(c)}(x - c)$$



Derivatives of inverse trig

derivative of arcsinx : $f'(x) = \frac{1}{\sqrt{1-x^2}}$

derivative of arccosx : $f'(x) = -\frac{1}{\sqrt{1-x^2}}$

derivative of arctanx : $f'(x) = \frac{1}{1+x^2}$

derivative of arccotx : $f'(x) = -\frac{1}{1+x^2}$

derivative of arcsecx : $f'(x) = \frac{1}{|x|\sqrt{x^2-1}}$

derivative of arccscx : $f'(x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Implicit differentiation

$f'(x)$ = derivative of $f(x)$ ← first derivative

$f''(x)$ = derivative of $f'(x)$ ← second derivative

Logarithmic differentiation

Derivative of $\log_a x$ $\quad f'(x) = \dfrac{1}{x \ln a}$

Derivative of a^x $\quad f'(x) = a^x(\ln a)$

Derivative of e^x $\quad f'(x) = e^x$

Derivative of $\ln x$ $\quad f'(x) = \dfrac{1}{x}$

1. Find equation(s) of the line(s) normal to the curve $y^2 - 7xy = -48$ at the point where $x = 2$

$2y(y') - 7y - 7x(y') = 0$

$y'(2y - 7x) = 7y$

$y' = \frac{7y}{2y - 7x}$

$y^2 - 7(2)y = -48$

$y^2 - 14y + 48$

$(y-8)(y-6)$

$y = 8 \quad y = 6$

$y'(2,8) = \frac{7(8)}{2(8) - 7(2)}$

$= 28 \rightarrow m = -\frac{1}{28}$

$\boxed{y - 8 = -\frac{1}{28}(x - 2)}$

$y'(2,6) = \frac{7(6)}{2(6) - 7(2)}$

$= -21 \rightarrow m = \frac{1}{21}$

$\boxed{y - 6 = \frac{1}{21}(x - 2)}$

2. Use logarithmic differentiation to find the derivative of: $y = \frac{e^{3x}\sqrt{x-1}}{(\cos x)(2x+4)^3}$

$\ln y = \ln e^{3x} + \ln (x-1)^{\frac{1}{2}} - \ln \left[(\cos x)(2x+4)^3\right]$

$\ln y = 3x + \frac{1}{2}\ln(x-1) - \ln(\cos x) - \ln(2x+4)^3$

$\frac{y'}{y} = 3 + \frac{1}{2}\left(\frac{1}{x-1}\right) - \left(\frac{-\sin x}{\cos x}\right) - 3\left(\frac{2}{2x+4}\right)$

$\frac{y'}{y} = 3 + \frac{1}{2x-2} + \tan x - \frac{3}{x+2}$

$y' = \left[\frac{e^{3x}\sqrt{x-1}}{(\cos x)(2x+4)^3}\right]\left[3 + \frac{1}{2x-2} + \tan x - \frac{3}{x+2}\right]$

3. Find the slope of the tangent to $y^4+x^2=15$ at $(-2,1)$ 10

$y^4+x^2=15$

$\frac{dy}{dx} y^4 + \frac{dy}{dx} x^2 = \frac{dy}{dx}$

$\frac{dy}{dx} 4y^3 + 2x = 0$

$\frac{dy}{dx} 4y^3 = -2x$

$\frac{dy}{dx} = \frac{-2x}{4y^3}$ $\qquad \frac{-2(-2)}{4(1)^3} = \frac{4}{4} = \boxed{1}$

Group #4 MVT, Rolles Theorem, Continuity, Differentiability, Kinematics

11

Mean Value Theorem

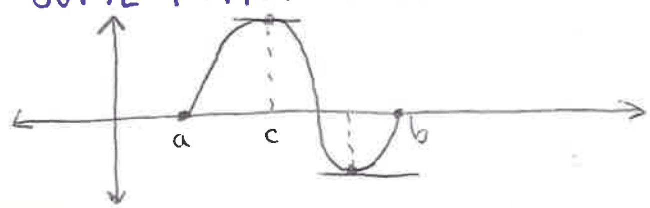
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Suppose a function is CONTINUOUS and DIFFERENTIABLE on the interval (a, b) for some number c.

Rolles Theorem

- you will be given a function
- you must prove that this function is continuous and differentiable.
- you will be given an interval
  ex: $[5, 5]$
- plug in both values into the function
- If these equal the same thing.

THE DERIVATIVE = 0 at SOME POINT on the interval



* to be differentiable, f(x) must be continuous!

Kinematics

Velocity: (v)

$$v(t) = \frac{ds}{dt} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

rate of change of distance

Average Velocity: $\frac{\text{change of } s}{\text{change of } t}$

Acceleration:

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

V	a	speed
+	+	increasing
+	−	decreasing
−	+	decreasing
−	−	increasing

v(t): (+) 4 (−) 5 (+)

Change direction at 4 s and 5 s.

Total Distance:

$$|S(0) - S(t_1)| + |S(t_1) - S(t_2)| + |S(t_2) - S(t_3)|$$

Continuity

If function f is differentiable at $x = c$, it is also continuous at $x = c$.

ex. 1) check if f is defined

$$f(x) = \begin{cases} \sin x, & x \geq 0 \\ x^2 + 5x, & x < 0 \end{cases}$$

$f(0) = \sin 0 = 0$

$f(0) = 0^2 + 5(0) = 0$ ✓ defined

2) check for continuity with limits

$\lim_{x \to 0^-} x^2 + 5x = 0^2 + 5(0) = 0$

$\lim_{x \to 0^+} \sin x = \sin 0 = 0$ ✓ continuous

Differentiability

function is differentiable if and only if it is differentiable at every value of x in its domain.

f is differentiable at $x = a$ if

$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ and

$f'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ both exist and are equal.

... 3) check for differentiability

$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} 2x + 5 = 2(0) + 5 = 5$

$\lim_{x \to 0^+} -\cos x = -\cos 0 = -1$ not differentiable

1. Given $f(x) = \begin{cases} x^a & \text{if } x \le 5 \\ -b(x+5)^2 & \text{if } x > 5 \end{cases}$ find a and b if $f(x)$ is differentiable for $x \in \mathbb{R}$

Solution attached

2. In the interval $[8, 20]$, find all values c that satisfy MVT for $f(x) = 3x^2$

Solution attached

3. A ball is thrown vertically upward at $t = 0$ and has height $h(t) = -16t^2 + 88t + 20$ ft.

a. Find ball's velocity and acceleration

b. What is max height attained by the ball and at what time

Solution attached.

Solutions

1. $f(5) = x^a = 5^a$

$$f(5) = b(5+5)^2 = b(10)^2 = 100b$$

$$5^a = 100b \qquad b = \frac{5^a}{100}$$

$$f'(x) = (x^a)' = \cancel{a(5)^{a-1}}\ a(x)^{a-1}$$

$$f'(x) = (b(x+5)^2)' = 2b(x+5)((x+5))' = 2b(x+5)$$

$$f'(5) = a(5)^{a-1}$$

$$f'(5) = 2b(5+5) = 20b$$

$$a(5)^{a-1} = \left(\frac{5^a}{100}\right)(20)$$

$$a(5)^{a-1} = \frac{5^a}{5}$$

$$a(5)^{a-1} = 5^{a-1}$$

$$\boxed{a = 1}$$

$$b = \frac{5^a}{100}$$

$$= \frac{5^1}{100}$$

$$\boxed{b = \frac{1}{20}}$$

2. $f(8) = 3(8)^2 = 192$

$f(20) = 1200$

$$f'(c) = \frac{1200-192}{20-8} = 84$$

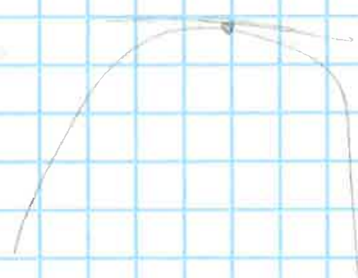
$f'(x) = 6x$

$84 = 6c$

$\boxed{c = 14}$

3. a. $v(t) = -32t + 88$

$a(t) = -32$



b. $v(t) = 0$ at max

$0 = -32t + 88$

$t = 2.75$ seconds

$h(2.75) = -16(2.75)^2 + 88(2.75) + 20$

$= 141$ ft