1st Semester Final Review Answers

I. Limits and Continuity

1\. a) $\lim\limits_{x\to 1} \dfrac{x^2-3x+2}{x^2-1} = \lim\limits_{x\to 1} \dfrac{(x-2)(x-1)}{(x+1)(x-1)} = \lim\limits_{x\to 1} \dfrac{x-2}{x+1} = \dfrac{1-2}{1+1} = \boxed{-\dfrac{1}{2}}$

b) $\lim\limits_{x\to 0} \dfrac{1-\sin x}{\cos^2 x} = \lim\limits_{x\to 0} \dfrac{1-\sin x}{1-\sin^2 x} = \lim\limits_{x\to 0} \dfrac{1-\sin x}{(1-\sin x)(1+\sin x)}$

$= \lim\limits_{x\to 0} \dfrac{1}{1+\sin x} = \lim\limits_{x\to 0} \dfrac{1}{1+\sin 0} = \dfrac{1}{1+0} = \boxed{1}$

c) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-2}{x-4} = \dfrac{\sqrt{2}-2}{2-4} = \boxed{\dfrac{\sqrt{2}-2}{-2}}$

2\. $f(x) = \begin{cases} ax^2+10 & x \le 2 \\ x^2-6x+b & x > 2 \end{cases}$ $\qquad f'(x) = \begin{cases} 2ax & x < 2 \\ 2x-6 & x > 2 \end{cases}$

Continuous

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$4a+10 = 4-12+b$

$4a+10 = -8+b$

$4a+18 = b$

$4\left(-\frac{1}{2}\right)+18 = b$

$16 = b$

Diff'able

$\lim\limits_{x\to 2^-} f'(x) = \lim\limits_{x\to 2^+} f'(x)$

$4a = 4-6$

$4a = -2$

$a = -\frac{1}{2}$

$\boxed{a = -\frac{1}{2},\ b = 16}$

II. Differentiation

1\. $f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt{5(x+h)}-\sqrt{5x}}{h} \cdot \dfrac{\sqrt{5(x+h)}+\sqrt{5x}}{\sqrt{5(x+h)}+\sqrt{5x}}$

$= \lim\limits_{h\to 0} \dfrac{5(x+h)-5x}{h\left(\sqrt{5(x+h)}+\sqrt{5x}\right)} = \lim\limits_{h\to 0} \dfrac{5\cancel{h}}{\cancel{h}\left(\sqrt{5(x+h)}+\sqrt{5x}\right)}$

$= \lim\limits_{h\to 0} \dfrac{5}{\sqrt{5(x+h)}+\sqrt{5x}} = \dfrac{5}{\sqrt{5x}+\sqrt{5x}} = \boxed{\dfrac{5}{2\sqrt{5x}}}$

II. 2. $f(x)=(x^3-2x^2)(x+2)$

$f(x) = x^4+2x^3-2x^3-4x^2$

$f(x) = x^4-4x^2$

$f'(x) = 4x^3-8x$

$f'(1) = 4(1)^3-8(1) = 4-8 = -4$

$f(1) = (1)^4-4(1)^2 = 1-4 = -3$

$\boxed{y+3 = -4(x-1)}$

3. $xy^2 + x^2y = 2$

$1\cdot y^2 + x\cdot 2y\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} = 0$

$y^2 + 2xy + (2xy + x^2)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y^2-2xy}{2xy+x^2}$

$x=1$

$1y^2 + 1y = 2$

$y^2+y-2=0$

$(y-1)(y+2)=0$

$y = 1, -2$

at $(1,1)$, $\frac{dy}{dx} = \frac{-1-2}{2+1} = -1 \Longrightarrow$ normal slope $m=1$

at $(1,-2)$, $\frac{dy}{dx} = \frac{-4-2(1)(-2)}{2(1)(-2)+1} = \frac{0}{-3} = 0 \Longrightarrow m=$ undefined

Normal Lines

$\boxed{y = 1(x-1)+1}$

$\boxed{x=1}$

4. a. $f(x) = \tan(2x^3)$

$\boxed{f'(x) = \sec^2(2x^3)(6x^2)}$

b. $f(x) = e^{4x^5-3x}\sqrt{\sin x}$

$\boxed{f'(x) = (20x^4-3)e^{4x^5-3x}\sqrt{\sin x} + e^{4x^5-3x}\cdot\frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)}$

5. a) $2x^3+3y^2+4 = 5x$

$6x^2 + 6y\frac{dy}{dx} = 5$

$\boxed{\frac{dy}{dx} = \frac{5-6x^2}{6y}}$

5. b. $e^{xy}+1 = x^3$

$e^{xy}\left(y + x\frac{dy}{dx}\right) = 3x^2$

$ye^{xy} + xe^{xy}\frac{dy}{dx} = 3x^2$

$\boxed{\frac{dy}{dx} = \frac{3x^2-ye^{xy}}{xe^{xy}}}$

II.

6. $y = \ln(5x^3 + 2)$

$$\frac{dy}{dx} = \frac{15x^2}{5x^3 + 2}$$

$$\frac{d^2y}{dx^2} = \frac{(5x^3+2)(30x) - (15x^2)(15x^2)}{(5x^3+2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{150x^4 + 60x - 225x^4}{(5x^3+2)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{60x - 75x^4}{(5x^3+2)^2}}$$

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7. $y = \cot x$

$y' = -\csc^2 x$

at $x = \frac{5\pi}{6} \rightarrow y' = -\csc^2\left(\frac{5\pi}{6}\right)$ $\quad y = \cot\left(\frac{5\pi}{6}\right)$

$y' = \frac{-1}{\left(-\frac{1}{2}\right)^2}$ $\quad y = -\sqrt{3}$

$y' = -4$

tangent line:

$$\boxed{y = -4\left(x - \frac{5\pi}{6}\right) - \sqrt{3}}$$

8. $f(x) = \cos^2 x$

$f'(x) = 2\cos x(-\sin x)$

$f'\left(\frac{\pi}{3}\right) = -2\left(\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{3}\right)$

$= -2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{\sqrt{3}}{2} \rightarrow$ Normal slope $= \frac{1}{2\sqrt{3}}$

$f\left(\frac{\pi}{3}\right) = \left(\cos\frac{\pi}{3}\right)^2$

$= \left(\frac{1}{2}\right)^2$

$= \frac{1}{4}$

normal line:

$$\boxed{y = \frac{2}{\sqrt{3}}\left(x - \frac{\pi}{3}\right) + \frac{1}{4}}$$

III. Applications of Derivatives

1. $f(x) = x^4 - 8x^2 + 12$ on $[-1, 2]$

$f'(x) = 4x^3 - 16x$

$0 = 4x(x^2 - 4)$

$x = 0, 2, -2$

f': $+$ on $(-1, 0)$, $-$ on $(0, 2)$

x	$f(x)$
-1	5
0	12
2	-4

$$\boxed{\begin{array}{l} \text{Abs max: } 12 \text{ at } x = 0 \\ \text{Abs min: } -4 \text{ at } x = 2 \end{array}}$$

2. $f(x) = x^3 + x - 4$ is continuous and differentiable for all real numbers.

$f'(x) = 3x^2 + 1$ $\quad f(-2) = -8 - 2 - 4 = -14, \quad f(3) = 27 + 3 - 4 = 26$

$$\frac{26 - (-14)}{3 - (-2)} = 3c^2 + 1$$

$$\frac{40}{5} = 3c^2 + 1$$

$$7 = 3c^2$$

$$\boxed{c = \pm\sqrt{\frac{7}{3}}}$$

3. $f(x) = 6x^5 - 15x^4 + 10x^3$

$f'(x) = 30x^4 - 60x^3 + 30x^2$

$0 = 30x^2(x^2 - 2x + 1)$

$0 = 30x^2(x-1)^2$

f': $+$ on $(-\infty, 0)$, $+$ on $(0, 1)$, $+$ on $(1, \infty)$

$f''(x) = 120x^3 - 180x^2 + 60x$

$0 = 60x(2x^2 - 3x + 1)$

$0 = 60x(2x - 1)(x - 1)$

f'': $-$ on $(-\infty, 0)$, $+$ on $(0, \frac{1}{2})$, $-$ on $(\frac{1}{2}, 1)$, $+$ on $(1, \infty)$

Increasing: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Decreasing: never

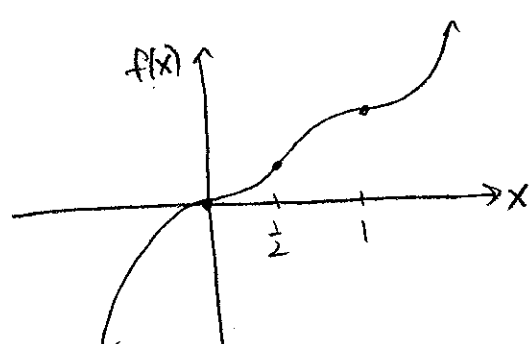
Crit. pts: $x = 0, 1$

Max/Min: None

Infl. Pts: $x = 0, \frac{1}{2}, 1$

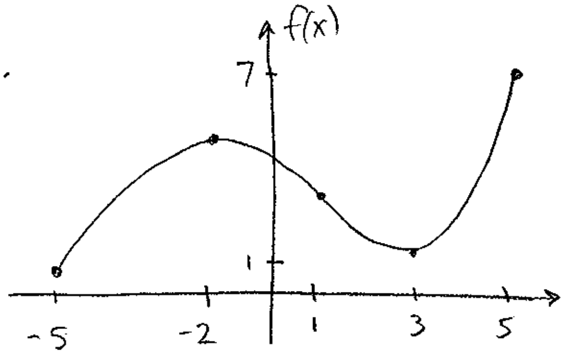
Concave Up: $(0, \frac{1}{2}) \cup (1, \infty)$

Concave Down: $(-\infty, 0) \cup (\frac{1}{2}, 1)$



III

4.



5. a. A, E

b. C

c. B, D

IV. 1. $s(t) = \frac{1}{3}t^3 - \frac{9}{2}t^2 + 18t + 2$

a. $v(t) = t^2 - 9t + 18$

$a(t) = 2t - 9$

b. $v(t) = (t-6)(t-3)$

v: + on (0, 3), − on (3, 6), + after 6

a: − before $\frac{9}{2}$, + after $\frac{9}{2}$

$\boxed{(3, \frac{9}{2}) \cup (6, \infty)}$

c. $t = 3, 6$

d. $s(0) = 2$

$s(3) = 24.5$

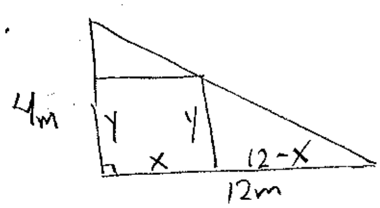
$s(5) = 21.16667$

22.5

3.33333

$\boxed{25.833 \text{ cm}}$

V. 1.



$A = xy$

$\frac{4}{y} = \frac{12}{12-x}$

$12y = 4(12-x)$

$y = \frac{1}{3}(12-x)$

$A = x\left(\frac{1}{3}(12-x)\right)$

$A = 4x - \frac{1}{3}x^2$

$A' = 4 - \frac{2}{3}x$

$0 = 4 - \frac{2}{3}x$

$4 = \frac{2}{3}x$

$6 = x$

A': + before 6, − after 6

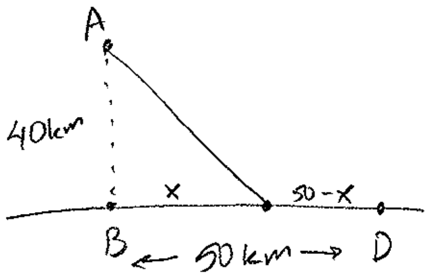
6 ← max

Max area: $A = 6 \cdot \frac{1}{3} \cdot 6$

$A = \boxed{12 \text{ m}^2}$

IV

2.



Desert 45 km/hr Road 75 km/hr

$$T = \frac{\sqrt{x^2+40^2}}{45} + \frac{50-x}{75}$$

$$\frac{dT}{dx} = \frac{1}{45} \cdot \frac{1}{2}\left(x^2+40^2\right)^{-\frac{1}{2}}(2x) - \frac{1}{75}$$

$$0 = \frac{x}{45\sqrt{x^2+40^2}} - \frac{1}{75}$$

$$0 = \frac{75x - 45\sqrt{x^2+40^2}}{45 \cdot 75\sqrt{x^2+40^2}}$$

$$0 = 75x - 45\sqrt{x^2+40^2}$$

$$\frac{5}{3}x = \sqrt{x^2+40^2}$$

$$\frac{25x^2}{9} = x^2 + 40^2$$

$$\frac{16x^2}{9} = 40^2$$

$x^2 = 900$

$x = 30, \cancel{-30}$

$\frac{dT}{dx}$: $+$ | -30 | $-$ | 30 | $+$

$30 \leftarrow$ Min

$x = 30$ km

$$\hookrightarrow T = \frac{\sqrt{30^2+40^2}}{45} + \frac{50-30}{75}$$

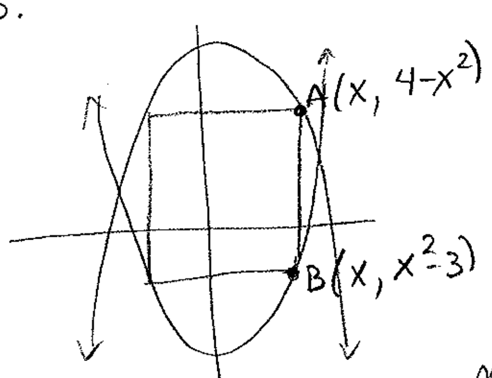
$$T = \frac{10}{9} + \frac{4}{15}$$

$T \approx 1.37778$ hours

≈ 82.67 minutes

She should travel to the point 30km from B between B and D. She wins the prize.

3.



$$A = 2x\left[(4-x^2) - (x^2-3)\right]$$

$$A = 2x(7-2x^2)$$

$$A = 14x - 4x^3$$

$$A' = 14 - 12x^2$$

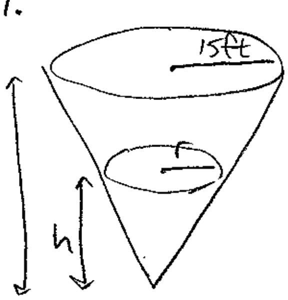
A': $-$ | $-\sqrt{\frac{7}{6}}$ | $+$ | $\sqrt{\frac{7}{6}}$ | $-$

Max area occurs when $x = \sqrt{\frac{7}{6}}$

$$A = 2\left(\sqrt{\tfrac{7}{6}}\right)\left(7 - 2\left(\sqrt{\tfrac{7}{6}}\right)^2\right)$$

$\boxed{A \approx 10.1}$

VI 1.



$\frac{dV}{dt} = 20 \text{ ft}^3/\text{min}$

what is $\frac{dh}{dt}$ when $h = 12$?

$\frac{r}{h} = \frac{15}{34}$

$r = \frac{15}{34} h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{\pi}{3}\left(\frac{15}{34}h\right)^2 h$

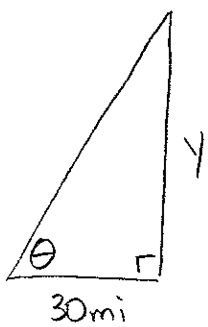
$V = \frac{75\pi}{1156} h^3$

$\frac{dV}{dt} = \frac{225\pi}{1156} h^2 \frac{dh}{dt}$

$20 = \frac{225\pi}{1156}(12)^2 \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} \approx 0.227 \text{ ft/min}}$

2.



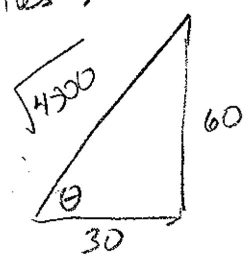
$\frac{dy}{dt} = 5600 \text{ mi/hr}$

what is $\frac{d\theta}{dt}$ when $y = 60$ miles?

$\tan\theta = \frac{y}{30}$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$

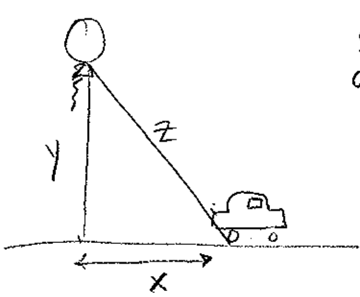
$(\sqrt{5})^2 \frac{d\theta}{dt} = \frac{1}{30}(5600)$



$\cos\theta = \frac{30}{\sqrt{4500}} = \frac{30}{30\sqrt{5}} = \frac{1}{\sqrt{5}}$

$\frac{d\theta}{dt} \approx 37.3 \text{ rad/hr} \longrightarrow \frac{37.3 \text{ rad/hr}}{3600 \text{ sec/hr}} \approx \boxed{0.0519 \text{ rad/sec}}$

3.



$\frac{dy}{dt} = 20 \text{ ft/sec}$ $\quad \frac{dx}{dt} = 40 \text{ ft/sec}$

what is $\frac{dz}{dt}$ when $x = 40$ ft and $y = 30$ ft?

$x^2 + y^2 = z^2$

$40^2 + 30^2 = z^2$

$z = 50$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$(40)(40) + (30)(20) = (50)\frac{dz}{dt}$

$\boxed{\frac{dz}{dt} = 44 \text{ ft/sec}}$

VII.

1. D $[5, \infty)$
   R $[-8, \infty)$

2. $\frac{4\pi}{3}, \frac{5\pi}{3}$

3. $g(3) = \frac{2}{3-5} = -1$

$f(-1-7) = f(-8) = 6(-8) = \boxed{-48}$

4. $15 = 12 + \sqrt{3x+1}$

$3 = \sqrt{3x+1}$

$9 = 3x+1$

$\boxed{\frac{8}{3} = x}$

5. $\log_2(x^2 - 2x) = 3$

$2^3 = x^2 - 2x$

$0 = x^2 - 2x - 8$

$0 = (x-4)(x+2)$

$x = 4, \cancel{-2}$ ← not in domain

$\boxed{x = 4}$

6. Amp: 5

Axis: $y = -3$

Shift: Right 1

Period: 8 $\Rightarrow$ $8 = \frac{2\pi}{B}$

$B = \frac{\pi}{4}$

$\boxed{y = 5\cos\left(\frac{\pi}{4}(x-1)\right) - 3}$