

Review – Differential Calculus 2 (HL)

◆ Paper 2 Review – 9 Questions ◆ calculator allowed [worked solutions included]

syllabus content: chain rule, product rule, quotient rule, 2nd derivative test for maxima and minima, tangent lines, inflexion points, optimization, implicit differentiation

- The point $P(1, p)$, where $p > 0$, lies on the curve $2x^2y + 3y^2 = 16$.
 - Calculate the value of p .
 - Calculate the gradient of the tangent to the curve at point P .
- Given that $y = \frac{x}{e^x} + \sqrt{2x}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
- The normal to the curve $y = \frac{k}{x} + \ln(x^2)$, for $x \neq 0$, $k \in \mathbb{R}$, at the point where $x = 2$, has the equation $3x + 2y = c$ where c is an unknown constant. Find the exact value of k .
- Find the **exact** coordinates of any maximum or minimum points on the graph of $y = \frac{1}{2} \sin 2x + \cos x$ in the interval $0 \leq x \leq 2\pi$. Clearly identify as a maximum or minimum.
- Calculate the minimum distance from the point $A\left(2, -\frac{1}{2}\right)$ to the parabola $y = x^2$.
- The function f is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.
 - Show that $f'(x) = \frac{2x - x^2 \ln 2}{2^x}$
 - Obtain an expression for $f''(x)$, simplifying your answer as far as possible.
 - Find the **exact** value of x satisfying the equation $f'(x) = 0$
 - Show that this value gives a maximum value for $f(x)$.
 - Find the x -coordinates of the two points of inflexion on the graph of f .
- Let $f(x) = \frac{x^2 + 5x + 5}{x + 2}$, $x \neq -2$
 - Find $f'(x)$
 - Solve $f'(x) > 2$
- Consider the curve with the equation $x^2 + xy + y^2 = 19$.
 - Find the equation of the line tangent to the curve at the point where $x = -2$ and $y > 0$.
 - Find the x -coordinate of any point(s) where the tangent to the curve is parallel to the y -axis.
- A rectangle is drawn so that its lower vertices are on the x -axis and its upper vertices are on the curve $y = e^{-x^2}$. The area of this rectangle is denoted by A .
 - Write down an expression for A in terms of x .
 - Find the maximum value of A .

Bonus: Find the exact value of the constant c in question 3.

Review – Differential Calculus 2 (HL)

Worked Solutions

1. (a) $2(1)^2 y + 3y^2 = 16 \Rightarrow 3y^2 + 2y - 16 = 0 \Rightarrow (3y+8)(y-2) = 0 \Rightarrow y = -\frac{8}{3}$ or $y = 2$

Since $p > 0$, then $p = 2$

(b) $2 \frac{d}{dx}(x^2 y) + 3 \frac{d}{dx}(y^2) = \frac{d}{dx}(16) \Rightarrow 2 \left(2x \cdot y + x^2 \cdot \frac{dy}{dx} \right) + 3 \left(2y \cdot \frac{dy}{dx} \right) = 0$

$$\frac{dy}{dx}(2x^2 + 6y) = -4xy \Rightarrow \frac{dy}{dx} = \frac{-4xy}{2x^2 + 6y}$$

At (1,2): $\frac{dy}{dx} = \frac{-4(1)(2)}{2(1)^2 + 6(2)} = \frac{-8}{14} = -\frac{4}{7}$

2. $y = \frac{x}{e^x} + \sqrt{2x} = xe^{-x} + (2x)^{1/2}$ $\frac{dy}{dx} = 1 \cdot e^{-x} + x \cdot e^{-x}(-1) + \frac{1}{2}(2x)^{-1/2}(2) = e^{-x} - x \cdot e^{-x} + (2x)^{-1/2}$

$$\frac{dy}{dx} = \frac{1-x}{e^x} + \frac{1}{\sqrt{2x}}$$

$$\frac{dy}{dx} = (1-x)e^{-x} + (2x)^{-1/2} \quad \frac{d^2y}{dx^2} = -1 \cdot e^{-x} + (1-x)e^{-x}(-1) + \left(-\frac{1}{2}\right)(2x)^{-3/2}(2)$$

$$\frac{d^2y}{dx^2} = -e^{-x} + (x-1)e^{-x} - (2x)^{-3/2}$$

$$\frac{d^2y}{dx^2} = -e^{-x} + xe^{-x} - e^{-x} - \frac{1}{\sqrt{(2x)^3}} = (x-2)e^{-x} - \frac{1}{\sqrt{8x^3}}$$

$$\frac{d^2y}{dx^2} = xe^{-x} - \frac{1}{2\sqrt{2x^3}}$$

3. $y = kx^{-1} + \ln(x^2) \Rightarrow \frac{dy}{dx} = -kx^{-2} + \frac{1}{x^2} \cdot 2x = -\frac{k}{x^2} + \frac{2}{x}$; when $x = 2$: $\frac{dy}{dx} = -\frac{k}{2^2} + \frac{2}{2} = \frac{-k+4}{4}$

$3x + 2y = c \Rightarrow y = -\frac{3}{2}x + \frac{c}{2}$; slope of normal is $\frac{2}{3}$, thus $\frac{-k+4}{4} = \frac{2}{3} \Rightarrow -k+4 = \frac{8}{3} \Rightarrow k = \frac{4}{3}$

Review – Differential Calculus 2 (HL)

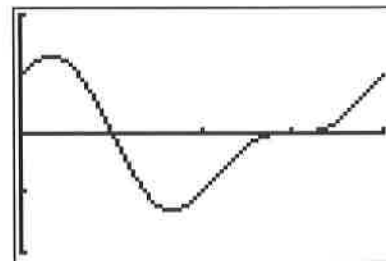
Worked Solutions (continued)

$$4. \frac{dy}{dx} = \frac{1}{2}(\cos 2x) \cdot 2 - \sin x = \cos 2x - \sin x; \frac{d^2y}{dx^2} = -2 \sin 2x - \cos x$$

substituting $1 - 2\sin^2 x$ for $\cos 2x$ gives:

$$1 - 2\sin^2 x - \sin x = 0 \Rightarrow 2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0 \Rightarrow \sin x = \frac{1}{2} \text{ or } \sin x = -1$$



graph of $y = \frac{1}{2} \sin 2x + \cos x$, $0 \leq x \leq 2\pi$

$$\blacksquare \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6} \quad y\left(\frac{\pi}{6}\right) = \frac{1}{2} \sin\left(2 \cdot \frac{\pi}{6}\right) + \cos \frac{\pi}{6} = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

$$y\left(\frac{5\pi}{6}\right) = \frac{1}{2} \sin\left(2 \cdot \frac{5\pi}{6}\right) + \cos \frac{5\pi}{6} = \frac{1}{2} \left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{4}$$

$$\text{at } x = \frac{\pi}{6}: \frac{d^2y}{dx^2} = -2 \sin\left(2 \cdot \frac{\pi}{6}\right) - \cos \frac{\pi}{6} = -2 \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} < 0; \text{ thus, there is a max at } x = \frac{\pi}{6}$$

$$\text{at } x = \frac{5\pi}{6}: \frac{d^2y}{dx^2} = -2 \sin\left(2 \cdot \frac{5\pi}{6}\right) - \cos \frac{5\pi}{6} = -2 \cdot \left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2} > 0; \text{ thus, a min at } x = \frac{5\pi}{6}$$

$$\blacksquare \sin x = -1 \Rightarrow x = \frac{3\pi}{2}$$

$$\text{at } x = \frac{3\pi}{2}: \frac{d^2y}{dx^2} = -2 \sin\left(2 \cdot \frac{3\pi}{2}\right) - \cos \frac{3\pi}{2} = -2 \cdot 0 - 0 = 0; \text{ 2}^{\text{nd}} \text{ derivative test is not conclusive}$$

$$\text{test sign of 1}^{\text{st}} \text{ derivative just before and after } x = \frac{3\pi}{2} \approx 4.71; \quad \frac{dy}{dx} = \cos 2x - \sin x$$

$$\text{at } x = 4.5: \frac{dy}{dx} = \cos(9) - \sin(4.5) \approx 0.0664 > 0; \text{ at } x = 5: \frac{dy}{dx} = \cos(10) - \sin(5) \approx 0.120 > 0$$

therefore, no max or min at $x = \frac{3\pi}{2}$

Therefore, maximum at $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{4}\right)$, and minimum at $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{4}\right)$

Review – Differential Calculus 2 (HL)

Worked Solutions (continued)

5. let d be the minimum distance from $\left(2, -\frac{1}{2}\right)$ to $f(x) = x^2$

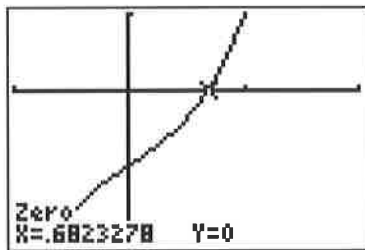
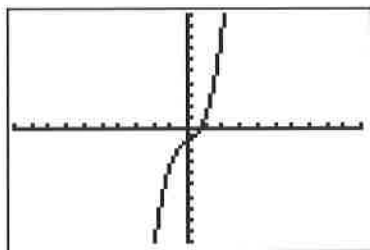
$$\begin{aligned} d &= \sqrt{(x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2} = \sqrt{x^2 - 4x + 4 + x^4 + x^2 + \frac{1}{4}} \\ &= \sqrt{x^4 + 2x^2 - 4x + \frac{17}{4}} \end{aligned}$$

The min value for the function $d = \sqrt{x^4 + 2x^2 - 4x + \frac{17}{4}}$

will occur where the function $x^4 + 2x^2 - 4x + \frac{17}{4}$ has a minimum.

$$y = x^4 + 2x^2 - 4x + \frac{17}{4} \Rightarrow \frac{dy}{dx} = 4x^3 + 4x - 4 = 0 \Rightarrow x^3 + x - 1 = 0$$

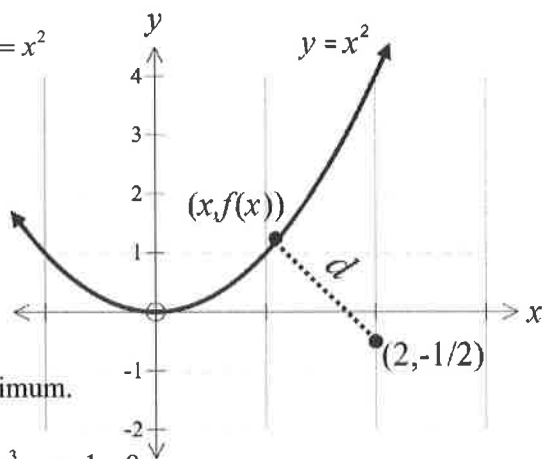
Solve $x^3 + x - 1 = 0$ by finding any x -intercept(s) of $y = x^3 + x - 1$



solution: $x \approx 0.6823278038\dots$

$$d = \sqrt{0.682^4 + 2(0.682)^2 - 4(0.682) + \frac{17}{4}} \approx 1.6335813\dots$$

Thus, the minimum distance from the point $\left(2, -\frac{1}{2}\right)$ to the graph of $y = x^2$ is approximately 1.63



Review – Differential Calculus 2 (HL)

Worked Solutions (continued)

$$6. (a) (i) f(x) = \frac{x^2}{2^x} \Rightarrow f'(x) = \frac{2^x \cdot 2x - x^2 \cdot 2^x \cdot \ln 2}{(2^x)^2} = \frac{2^x (2x - x^2 \ln 2)}{2^x \cdot 2^x} = \frac{2x - x^2 \ln 2}{2^x} \quad \text{Q.E.D.}$$

$$(ii) f''(x) = \frac{2^x (2 - 2x \ln 2) - (2x - x^2 \ln 2) 2^x \ln 2}{(2^x)^2} = \frac{2^x (2 - 2x \ln 2 - 2x \ln 2 + x^2 (\ln 2)^2)}{2^x \cdot 2^x}$$

$$= \frac{2 - 4x \ln 2 + x^2 (\ln 2)^2}{2^x} = \frac{(\ln 2)^2 x^2 - (4 \ln 2)x + 2}{2^x}$$

$$(b) (i) f'(x) = \frac{2x - x^2 \ln 2}{2^x} = 0 \Rightarrow 2x - x^2 \ln 2 = 0 \Rightarrow x(2 - x \ln 2) = 0 \Rightarrow \cancel{x=0} \text{ since } x > 0$$

$$\text{OR } 2 - x \ln 2 = 0 \Rightarrow x = \frac{2}{\ln 2}$$

$$(ii) f''\left(\frac{2}{\ln 2}\right) = \frac{(\ln 2)^2 \left(\frac{2}{\ln 2}\right)^2 - (4 \ln 2) \left(\frac{2}{\ln 2}\right) + 2}{2^{\frac{2}{\ln 2}}} = \frac{4 - 8 + 2}{e^2} = -\frac{2}{e^2}$$

note: $2^{\frac{2}{\ln 2}} = \left(2^{\frac{1}{\ln 2}}\right)^2$; let $a = 2^{\frac{1}{\ln 2}} \Rightarrow a^{\ln 2} = 2 \Rightarrow \log_a 2 = \ln 2 \Rightarrow a = e \Rightarrow \left(2^{\frac{1}{\ln 2}}\right)^2 = e^2$

since $f''\left(\frac{2}{\ln 2}\right) = -\frac{2}{e^2} < 0$ graph of f is concave down at $x = \frac{2}{\ln 2} \Rightarrow$ max value for f at $x = \frac{2}{\ln 2}$

$$(c) \text{ solving } f''(x) = 0 \quad \text{polyRoots}\left(\left(\ln(2)\right)^2 \cdot x^2 - 4 \cdot \ln(2) \cdot x + 2, x\right)$$

$$\{0.845111188584, 4.92566897497\}$$

finding sign (pos. or neg.) of three 'test' points:

$$x = 0.8, x = 3 \text{ and } x = 5$$

since sign of $f''(x)$ changes at $x \approx 0.845$ and

at $x \approx 4.93$, the graph of f has points of

inflection at $x \approx 0.845$ and $x \approx 4.93$

$$f(x) = \frac{(\ln(2))^2 \cdot x^2 - 4 \cdot \ln(2) \cdot x + 2}{2^x}$$

Done

$$f(0.8) \quad 0.051358$$

$$f(3) \quad -0.249211$$

$$f(5) \quad 0.004637$$

Review – Differential Calculus 2 (HL)

Worked Solutions (continued)

$$7. (a) f'(x) = \frac{(x+2)(2x+5) - (1)(x^2+5x+5)}{(x+2)^2} = \frac{2x^2+9x+10-x^2-5x-5}{(x+2)^2} \Rightarrow f'(x) = \frac{x^2+4x+5}{(x+2)^2}$$

$$(b) \frac{x^2+4x+5}{(x+2)^2} > 2 \Rightarrow \frac{x^2+4x+5}{(x+2)^2} - 2 > 0 \Rightarrow \frac{x^2+4x+5-2(x+2)^2}{(x+2)^2} > 0 \Rightarrow$$

$$\frac{x^2+4x+5-2x^2-8x-8}{(x+2)^2} > 0 \Rightarrow \frac{-x^2-4x-3}{(x+2)^2} > 0 \Rightarrow \frac{x^2+4x+3}{(x+2)^2} < 0 \quad \text{Switch inequality sign after multiplying through by } -1$$

denominator $(x+2)^2$ is always positive; thus, $\frac{x^2+4x+3}{(x+2)^2} < 0$ when $x^2+4x+3 < 0$

$$(x+3)(x+1) < 0 \Rightarrow \begin{array}{c} \xleftarrow{-3} \quad \xrightarrow{-1} \\ \text{pos.} \quad | \quad \text{neg.} \quad | \quad \text{pos.} \end{array} \quad \text{and } f'(x) = \frac{x^2+4x+5}{(x+2)^2} \text{ undefined at } x = -2$$

Therefore, $f'(x) > 2$ when $-3 < x < -1$, $x \neq -2$

$$8. (a) x = -2: (-2)^2 - 2y + y^2 = 19 \Rightarrow y^2 - 2y - 15 = 0 \Rightarrow (y-5)(y+3) = 0 \Rightarrow y = 5 \text{ since } y > 0$$

$$\text{Thus, point of tangency is } (-2, 5); \text{ find } \frac{dy}{dx}: 2x + y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x-y}{x+2y}$$

$$\text{at } (-2, 5): \frac{dy}{dx} = \frac{-2(-2)-5}{-2+2 \cdot 5} = -\frac{1}{8}; \text{ eqn of tangent line: } y-5 = -\frac{1}{8}(x-(-2)) \Rightarrow y = -\frac{1}{8}x + \frac{19}{4}$$

(b) “parallel to the y -axis” \Rightarrow vertical line; slope is undefined \Rightarrow find where $\frac{dy}{dx}$ is undefined

$$\frac{dy}{dx} = \frac{-2x-y}{x+2y} \text{ is undefined when } x+2y=0 \Rightarrow y = -\frac{1}{2}x; \text{ substitute into original equation}$$

$$x^2 + x\left(-\frac{1}{2}x\right) + \left(-\frac{1}{2}x\right)^2 = 19 \Rightarrow x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2 = 19 \Rightarrow \frac{3}{4}x^2 = 19 \Rightarrow x^2 = \frac{76}{3}$$

Therefore, tangent to the curve is vertical when $x = \pm\sqrt{\frac{76}{3}}$ $\left[\text{or } x = \pm 2\sqrt{\frac{19}{3}}, \text{ or } x \approx \pm 5.03 \right]$

Review – Differential Calculus 2 (HL)

Worked Solutions (continued)

9. (a) $A(x) = 2x \cdot e^{-x^2}$

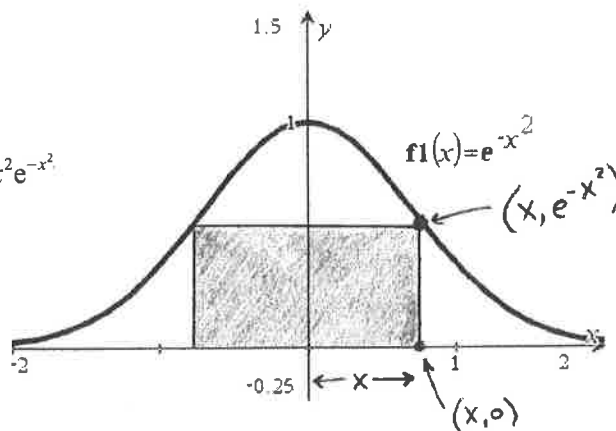
(b) $A'(x) = 2e^{-x^2} + 2x(e^{-x^2}(-2x)) = 2e^{-x^2} - 4x^2e^{-x^2}$

$$= (2 - 4x^2)e^{-x^2} = 0$$

$$e^{-x^2} \neq 0 \quad 2 - 4x^2 = 0 \Rightarrow x = \pm \sqrt{\frac{1}{2}}$$

but $x > 0$, so $x = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$; maximum area occurs when $x = \frac{\sqrt{2}}{2}$

$$\text{maximum area is } A\left(\frac{\sqrt{2}}{2}\right) = 2\left(\frac{\sqrt{2}}{2}\right) \cdot e^{-\left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{2}e^{-1/2} = \frac{\sqrt{2}}{\sqrt{e}} = \sqrt{\frac{2}{e}} \approx 0.858 \text{ units}^2$$



Bonus:

Since $k = \frac{4}{3}$ then $y = \frac{4}{3x} + \ln(x^2)$ which intersects the normal line $y = -\frac{3}{2}x + \frac{c}{2}$ at $x = 2$

$y = \frac{4}{3 \cdot 2} + \ln(2^2) = \frac{2}{3} + \ln 4$; thus the point $\left(2, \frac{2}{3} + \ln 4\right)$ must be on the normal line

$$\frac{2}{3} + \ln 4 = -\frac{3}{2} \cdot 2 + \frac{c}{2} \Rightarrow \frac{c}{2} = \frac{2}{3} + \ln 4 + 3 \Rightarrow c = \frac{4}{3} + 6 + 2 \ln 4 \Rightarrow c = \frac{22}{3} + \ln 16$$