2nd period

Final Review Notes

2016 ~ 2017

Chapter 17: Limits

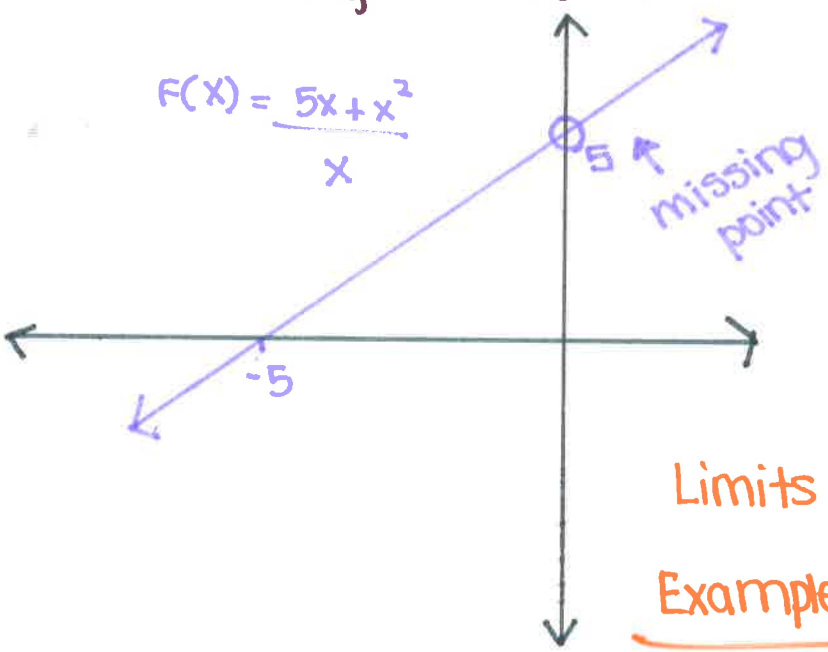
Definition: $\lim_{x \to a} F(x) = A$

1) As $F(x)$ converges to A, x approaches a.
2) Limits still exist at points of discontinuity.

Conditions to satisfy:

1) $F(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists
3) $F(a) = \lim_{x \to a} F(x)$

* A function f is said to be continuous at $x = a$, if and only if these 3 conditions are all satisfied:



* $\lim_{x \to 0} \frac{5x + x^2}{x} = 5$

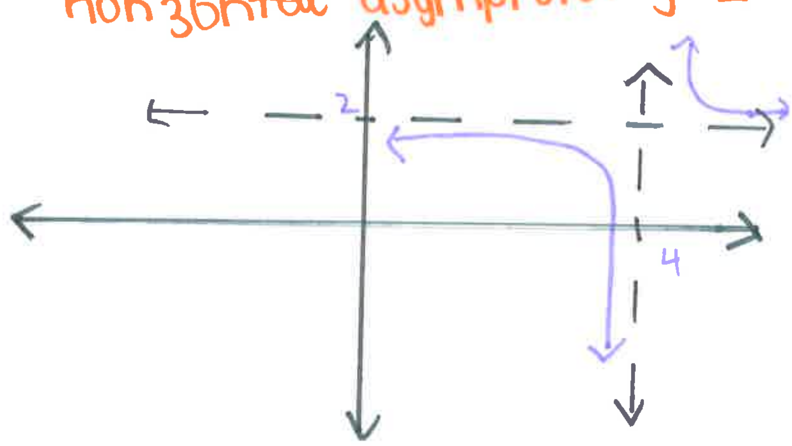
" the limit as x approaches 0, of $F(x) = \frac{5x+x^2}{x}$ is 5 "

Limits at ∞:

Example 1) $\lim_{z \to \infty} \frac{2x+3}{x-4} = \boxed{2}$

vertical asymptote: $x = 4$

horizontal asymptote: $y = 2$



Example 1) $\lim_{x \to 3} \frac{x^2-9}{x-3}$

$\lim_{x \to 3} \frac{x^2-9}{x-3}$

$= \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)}$

$= \lim_{x \to 3} (x+3)$

$= 6$

Differentiation from first principles:

derivative of $y = f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example i) Use the first principles formula $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the instantaneous rate of change in $f(x) = x^2 + 2x$ at the point where $x = 5$.

Solutions: $f(5) = 5^2 + 2(5) = 35$

$$\therefore f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h \to 0} \frac{(5+h)^2 + 2(5+h) - 35}{h}$$

$$= \lim_{h \to 0} \frac{25 + 10h + h^2 + 10 + 2h - 35}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 12h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h+12)}{\cancel{h}} \quad (\text{as } h \neq 0)$$

$$= 12$$

$\therefore$ rate of change in $f(x)$ at $x = 5$ is 12.

Trig Limits:

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{6x^2}$$

Solution

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{6x^2}$$

$$\lim_{x \to 0} \frac{\sin^2 x}{6x^2}$$

$$\lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{6x}$$

$$\lim_{x \to 0} 1\left(\frac{\sin x}{6x}\right)$$

$$\lim \frac{\sin x}{x} \cdot \frac{1}{6}$$

$$\lim_{x \to 0} 1\left(\frac{1}{6}\right)$$

$$\boxed{\lim_{x \to 0} \frac{1}{6}}$$

CHAPTER 18 : DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

{RULES OF DIFFERENTIATION}

- Differentiating a constant

  $f(x) = c$ (a constant) $\longrightarrow f'(x) = 0$

- Power rule

  $f(x) = x^n \longrightarrow f'(x) = nx^{n-1}$

- Chain rule

  $y = g(u)$ where $u = f(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

- Quotient rule

  $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$

- Product rule

  $f(x) = u(x)v(x)$ then $f'(x) = u'(x)v(x) + u(x)v'(x)$

{DIFFERENTIABILITY AND CONTINUITY}

- Continuity : If function f is differentiable at $x = c$, then f is continuous at $x = c$

f is differentiable at $x = a$ if $f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ [right-hand derivative]

and $f'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ [left-hand derivative] both exist and are equal.

Group 2

CHAPTER 18 — examples

① $f(x) = \dfrac{8x^8 - 6x^2}{\sqrt{x}}$ [quotient rule]

$$f'(x) = \frac{\sqrt{x}\,(64x^7 - 12x) - \frac{1}{2}x^{-\frac{1}{2}}(8x^8 - 6x^2)}{x}$$

② $f(x) = (2x-5)^3(5x^2-1)^4$ [chain and product rule]

$$f'(x) = 3(2x-5)^2(2)(5x^2-1)^4 + 4(5x^2-1)^3(10x)(2x-5)^3$$

③ Find the value of constants a and b ~~what~~ that make piecewise function $f(x)$ differentiable at $x=2$.

$$f(x) = \begin{cases} ax^3 & x \le 2 \\ b(x-3)^2 + 10 & x > 2 \end{cases}$$

$$\lim_{x\to 2^+} b(x-3)^2 + 10 = \lim_{x\to 2^-} ax^3$$

$$b(2-3)^2 + 10 = a(2)^3$$

$$b + 10 = 8a$$

$$b = 8\left(\tfrac{5}{7}\right) - 10$$

$$b = \frac{40}{7} - 10$$

$$\boxed{b = -\frac{30}{7}}$$

↳ Take derivative of the functions

$$\lim_{x\to 2^+} 2b(x-3) = \lim_{x\to 2^+} 3ax^2$$

$$-2b = 12a$$

$$-2(8a-10) = 12a$$

$$-16a + 20 = 12a$$

$$20 = 28a$$

$$\boxed{\frac{5}{7} = a}$$

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Group #3

Trig. Log. Differentiation
- using logarithms to differentiate

Example:
Find $\frac{dy}{dx}$ by logarithmic differentiation for $y = \frac{e^{2x}\sqrt{x-3x^2}}{(\sin x)(5x^2+x)^7}$.

1. $y = \frac{e^{2x}\sqrt{x-3x^2}}{(\sin x)(5x^2+x)^7}$

take log of both sides

$\ln y = \ln\left(\frac{e^{2x}\sqrt{x-3x^2}}{(\sin x)(5x^2+x)^7}\right)$

* log properties

$\ln y = \left(\ln(e^{2x}\sqrt{x-3x^2})\right) - \left(\ln(\sin x)(5x^2+x)^7\right)$

$\ln y = \cancel{\ln e^{2x}} + \ln(\sqrt{x-3x^2}) - \left(\ln(\sin x) + \ln(5x^2+x)^7\right)$

$\ln y = 2x + \frac{1}{2}\ln(x-3x^2) - \ln(\sin x) - 7\ln(5x^2+x)$

* differentiate

$\frac{1}{y}\cdot\frac{dy}{dx} = 2 + \frac{1(1-6x)}{2(x-3x^2)} - \frac{\cos x}{\sin x} - \frac{7(10x+1)}{5x^2+x}$

$\Rightarrow \boxed{\frac{dy}{dx} = y\left(2 + \frac{1-6x}{2x-6x^2} - \frac{\cos x}{\sin x} - \frac{70x+7}{5x^2+x}\right)}$

- differentiation of logs

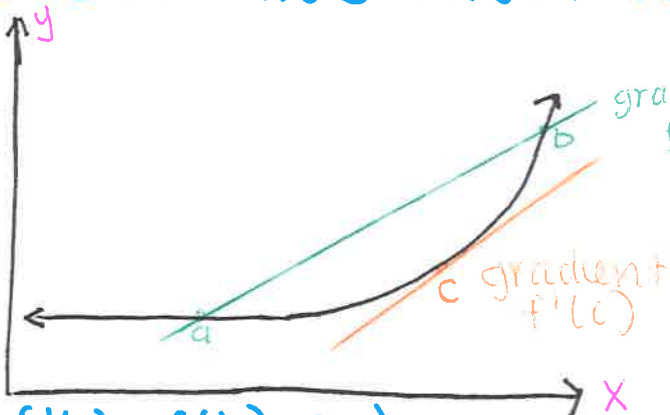
- $\frac{dy}{dx}$ of $\log_a x = \frac{1}{x\ln a}$

$\frac{dy}{dx}$ of $\ln x = \frac{1}{x}$

Example:
find $\frac{dy}{dx}$ of $\log_3(8x^2-7x)$

1. $\log_3(8x^2-7x)$
2. $\boxed{\frac{16x-7}{\ln_3(8x^2-7x)}}$ derivative of $8x^2-7x$

Mean Value Theorem



gradient $\frac{f(b)-f(a)}{b-a}$

- continuous in closed interval
- differentiable at open interval

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Example

Find all numbers c in the interval $[0,16]$ that satisfy MVT for $f(x) = 12\sqrt[4]{x}$

$a = 0$
$b = 16$

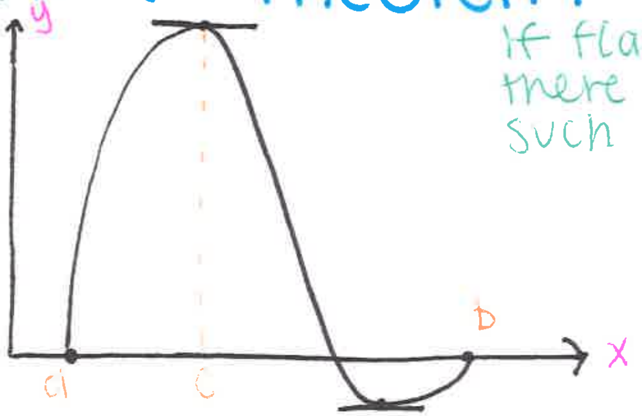
$f(x) = 12x^{1/4}$
$(12)'(x^{1/4}) + (12)(x^{1/4})'$
$+\ 12 \cdot \frac{1}{4}(x^{-3/4})$
$\frac{3}{\sqrt[4]{x^3}} = f'(c)$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{3}{\sqrt[4]{x^3}} = \frac{12\sqrt[4]{16} - 12\sqrt[4]{0}}{16-0} \Rightarrow \frac{3}{\sqrt[4]{x^3}} = \frac{24-0}{16}$

$\frac{3}{\sqrt[4]{x^3}} = 1.5$
$3 = 1.5 \cdot x^{3/4}$
$2 = x^{3/4}$
$x = \sqrt[3]{16}$
$\hookrightarrow c = \sqrt[3]{16}$

Rolle's Theorem



If $f(a) = f(b) = 0$, then there exists $c \in \,]a,b[$ such that $f'(c) = 0$

Example

Given $f(x) = x^2 - 3x + 2$. Use Rolle's Theorem show that $f'(x) = 0$ exist between $(1,2)$

If $f(1) = f(2)$ then $f'(c) = 0$ exists at $(1,2)$
$a = 1$
$b = 2$

$\Rightarrow f(1) = 1 - 3 + 2$
0

$\Rightarrow f(2) = 4 - 6 + 2$
0

$\therefore f'(c) = 0$ exists $(1,2)$

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Kinematics

- Displacement (position of an object)

  $s(t)$: displacement function
  - vector quantity
  - magnitude is distance from origin
  - sign indicates direction from origin.

on horizontal axis through 0
- $s(t) > 0$ P is right of 0
- $s(t) = 0$ P is at 0
- $s(t) < 0$ P is left of 0

- Velocity (average velocity is the gradient of a chord through $(t_1, s(t_1))$ & $(t_2, s(t_2))$.)

  $v(t) = s'(t)$ average velocity $= \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$

- Acceleration (~~rather~~ ratio of the change in velocity to the time taken)

  $a(t) = v'(t)$ average acceleration $= \dfrac{v(t_2) - v(t_1)}{t_2 - t_1}$

$s(0), v(0), a(0)$ give initial position, velocity, acceleration

Example

A particle moves in a straight line so that its position after t seconds is ~~$s(t) = \frac{1}{3}t^2$~~

$s(t) = \frac{1}{3}t^3 - \frac{11}{2}t^2 + 24t + 1$, $t \geq 0$

a. write expressions for the velocity and the acceleration of the particle in terms of t.

$v(t) = t^2 - 11t + 24 \ \frac{cm}{s}$

$a(t) = 2t - 11 \ \frac{cm}{s^2}$

Inverse Trig: $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Example: $y = \sqrt[5]{\cos^{-1}(2x^2)}$

$y = \cos^{-1}(2x^2)^{\frac{1}{5}}$

$\frac{dy}{dx} = \frac{1}{5}(\cos^{-1}(2x^2))^{-\frac{4}{5}}\left(\frac{-4x}{\sqrt{1-4x^4}}\right)$

Normal / Tangent Lines: Tangent $= y - f(c) = f'(c)(x-c)$
Normal $= y - f(c) = -\frac{1}{f'(c)}(x-c)$

Example: Find equation of normal to the graph of $f(x) = \sqrt{5x}$ at $x=5$

$x = 5$ $f(x) = 5$ $f'(x) = \frac{5}{2}(5x)^{-\frac{1}{2}}$
$x=5$

$y-5 = -2(x-5)$ $f'(x) = \frac{1}{2}$ slope is -2

Find the equations of tangents to $y=x^2$ from external point $(2,3)$

$\frac{dy}{dx} = 2x$ $x=a \rightarrow 2(a) = 2a$

equation of tangent $y - a^2 = 2a(x-a)$
must pass through $(2,3) \rightarrow 3 - a^2 = 2a(2-a)$

$3 - a^2 = 4a - 2a^2$

$a^2 - 4a + 3 = 0$

$a = 1, 3$

$y = 2x - 1$

$y = 6x - 9$

Per 2

Group 4 Curve Analysis

Tangents & Normals

- Slope of a tangent = derivative @ a point
- Slope of normal = $-\frac{1}{\text{derivative}}$

Max/Min

- Set deritives to 0
- Solve!
- Plot numbers on sign diagram to determine max/min.

Increasing/Decreasing Functions

- $f(x)$ is increasing for $f'(x) \geq 0$
- $f(x)$ is strictly increasing for $f'(x) > 0$
- $f(x)$ is decreasing for $f'(x) \leq 0$
- $f(x)$ is strictly decreasing for $f'(x) < 0$

If $f(x)$ is either increasing/decreasing for $x \in \mathbb{R}$, the function is monotone increasing/monotone decreasing

Stationary Points

- A stationary point is a point where $f'(x) = 0$ and can be local max, local min, or stationary inflection
  - Use a sign diagram to determine which

Inflections

- Point on a curve at which there is a change of curvature/shape.
- Set $f''(x)$ to 0
- Solve
- Plug into sign diagram

Stationary if $f'(x) = 0$
non-stationary if $f'(x) \neq 0$
*where x=point of inflection

concave down if $f''(x) \leq 0$

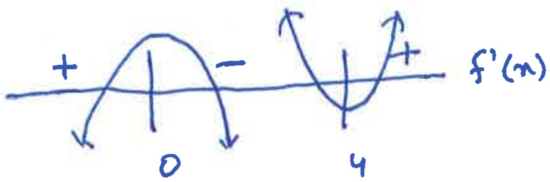
concave up if $f''(x) \geq 0$

Practice Problems

1) Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \le x \le 5$.

Solution:

$$\frac{dy}{dx} = 3x^2 - 12x = 3x(x-4) = 0$$



$\rightarrow f'(x)$ sign diagram

$x = 0 \longrightarrow$ local max

$x = 4 \longrightarrow$ local min

2) $y = 2x$ is tangent to the curve at $y = x^3 + ax + b$ at $x = 1$
find a and b.

Solution.

$y' = 2$ $\quad y' = 3x^2 + a = 2$ at $x = 1$

$y' = 3 + a = 2$

$a = -1$

$y = 2$ $\quad y = 1^3 + -1(1) + b = 2$

$b = 2$

$$\boxed{\begin{aligned} a &= -1 \\ b &= 2 \end{aligned}}$$

3) Find the equation of the tangent of $y = -2x^2$ at the point where $x = -1$

$y' = -2(2)x = -4x = -4(-1) = 4$

$y(-1) = -2(-1)^2 = -2$

$y = 4x + b \quad (-1, -2)$

$-2 = -4 + b$

$b = 2$

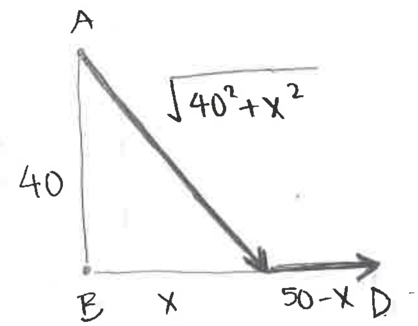
$$\boxed{y = 4x + 2}$$

CHAPTER 20: OPTIMIZATION

Group # 6

A dune buggy is on the desert at point A located 40 km from a point B, which lies on a straight road. The driver can travel at 45 km/hr on the desert and 75 km/hr on the road. What route should she travel to minimize travel time to point D (50 km from B)?

① Draw a diagram of given situation with appropriate notation



② Construct a formula with the variable to be optimized as the subject.

total distance = $\sqrt{40^2+x^2} + 50 - x$

time: $t(x) = \dfrac{\sqrt{40^2+x^2}}{45} + \dfrac{50-x}{75}$ ← distance / ← speed

$\left(\text{speed} = \dfrac{\text{distance}}{\text{time}} \quad \text{time} = \dfrac{\text{distance}}{\text{speed}} \right)$

③ Find the first derivative and solve for x which makes the first derivative zero.

$\dfrac{dt}{dx} = 0$

$t = \frac{1}{45}\sqrt{40^2+x^2} + \frac{1}{75}(50-x)$

$\dfrac{dt}{dx} = \frac{1}{45}\left(\cancel{\tfrac{1}{2}}\right)(40^2+x^2)^{-\frac{1}{2}}(\cancel{2}x) + \frac{1}{75}(-1) = 0$

$\dfrac{x}{\underset{3}{\cancel{45}}\sqrt{40^2+x^2}} = \dfrac{1}{\underset{5}{\cancel{75}}}$

$5x = 3\sqrt{40^2+x^2}$

$25x^2 = 9(40^2+x^2)$

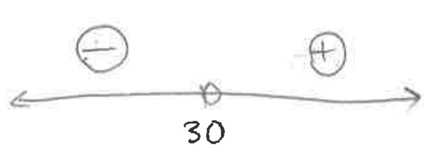
$25x^2 = 9x^2 + 14{,}400$

$16x^2 = 14{,}400$

$x^2 = 900 \longrightarrow x = \pm 30 \longrightarrow \boxed{x = 30 \text{ km}}$

(can't be negative)

④ Confirm the solution is max or min by 2nd derivative test or the first derivative sign diagram.



Related Rates

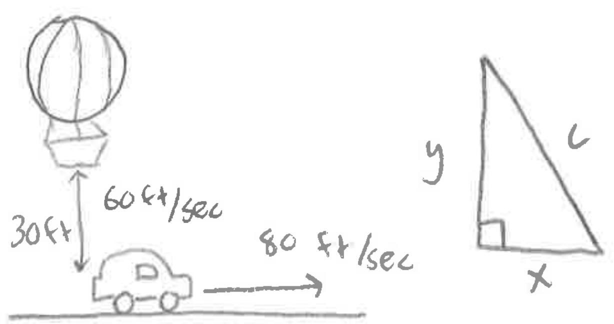
Group #6

Problem-Solving Steps:

1. Draw a large, clear diagram of the situation.
2. Write down the information, label the diagram, and make a distinction between variables and constants.
3. Write an equation connecting the variables.
4. Differentiate the equation with respect to t to obtain a differential equation.
5. Solve for the particular case which is at some instant in time.

Example 1:

A hot air balloon rises vertically at a rate of 60 ft/sec. A car travelling at 80 ft/sec drives directly under the balloon. The back of the car is directly below the balloon when the balloon is 30 ft above the ground. How fast is the distance between the balloon and car changing exactly one second after the back of the car is directly below the balloon.



$\frac{dy}{dt} = 60$ ft/sec

$\frac{dx}{dt} = 80$ ft/sec

$\frac{dc}{dt} = ?$

$t = 1$ sec

$y = 30$ ft

@ 1 sec
$x = 80$ ft
$y = 90$ ft

@ 1 second,
Balloon height: 30 ft + 60 ft/sec (1 sec) = 90 ft
Car distance: 0 ft + 80 ft/sec (1 sec) = 80 ft

$$x^2 + y^2 = c^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2c\frac{dc}{dt}$$

$$2(80)(80) + 2(90)(60) = 2(120.416)\frac{dc}{dt}$$

$$12{,}800 + 10{,}800 = 240.832\frac{dc}{dt}$$

$$\boxed{98.0 \text{ ft/sec} = \frac{dc}{dt}}$$

$$\sqrt{80^2 + 90^2} = c$$

$$c \simeq 120.416 \text{ ft}$$

* c = distance between balloon and car

Chapter 14: Geneva · Ian · Jonathan · Stefan

ot Product: $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

oss Product: $a \times b = (a_2 b_3 - a_3 b_2)i - (a_1 b_3 - a_3 b_1)j + (a_1 b_2 - a_2 b_1)k$

$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{matrix} i(a_2 b_3 - a_3 b_2) \\ -j(a_1 b_3 - a_3 b_1) \\ k(a_1 b_2 - a_2 b_1) \end{matrix}$

arallel Vectors: $\vec{a}$ and $\vec{b}$ are parallel if . . .

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = x \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

nit Vector: $1 = \dfrac{\vec{a}\text{ vector}}{|a|}$, vector divided by magnitude

llinear Vectors: $\vec{a}$, $\vec{b}$, and $\vec{c}$ are collinear if they are all parallel

$$\text{os } \theta = \frac{w \cdot v}{|w||v|}$$ and $\underbrace{\sin\theta |a||b| = |a \times b|}$

Magnitude of cross product is equal to the area of

* when dot product is 0, vectors are perpendicular
* when cross product is 0,

Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

$$-\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = -3i - 2j + k \qquad \left| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right| = \sqrt{14}$$

$$\frac{-3i - 2j + k}{\sqrt{14}} \cdot 5 = \boxed{-\frac{15}{\sqrt{14}} i - \frac{10}{\sqrt{14}} j + \frac{5}{\sqrt{14}} k}$$

Find m and n if $\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix}$ are parallel vectors.

$$\begin{pmatrix} 3 \\ m \\ n \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} =$$

$3 \rightarrow -12$

$\boxed{\cdot -4}$

$m(-4) = -20 \qquad n(-4) = 2$

$m = 5 \qquad n = -\frac{1}{2}$

Find t given that $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix}$ are perpendicular.

$$\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix} \cdot \begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix} = 0$$

$2t - \cancel{t^2}$
$+ \quad 12$
$+ \cancel{t^2} + t$
$= 0$

$3t + 12 = 0$

$3t = -12$

$t = -4$