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CHAPTER 17

LIMITS AND CONTINUITY

HOW TO KNOW IF A FUNCTION IS CONTINUOUS AT X = A?

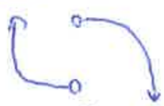
1) f(a) exists

2) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

3) $f(a) = \lim_{x \to a} f(x)$

POS/NEG SIDE LIMITS

$\lim_{x \to c} f(x)$ → both sides



DNE

$\lim_{x \to c^+} f(x)$ → right side

$\lim_{x \to c^-} f(x)$ → left side

WAYS TO SOLVE LIMITS

#1 FACTORING

$$\lim_{x \to 6} \frac{x^2 - 11x + 30}{x^2 - 36}$$

$$\lim_{x \to 6} \frac{(x-5)\cancel{(x-6)}}{(x+6)\cancel{(x-6)}} \quad \text{[cancel out } (x-6)]$$

$$\lim_{x \to 6} \frac{x-5}{x+6} \quad \text{[plug in]}$$

$$\lim_{x \to 6} \frac{(6-5)}{(6+6)} = \boxed{\frac{1}{12}}$$

#2 CONJUGATE

$$\lim_{x \to 4} \frac{\sqrt{x+3} - \sqrt{7}}{x-4}$$

$$\lim_{x \to 4} \frac{\sqrt{x+3} - \sqrt{7}}{x-4} \cdot \frac{(\sqrt{x+3} + \sqrt{7})}{(\sqrt{x+3} + \sqrt{7})}$$

$$\lim_{x \to 4} \frac{x+3-7}{\cancel{x-4}(\sqrt{x+3} + \sqrt{7})} = \boxed{\frac{1}{2\sqrt{7}}}$$

#3 $\boxed{\frac{\sin x}{x} = 1}$

$$\lim_{x \to 0} \frac{\sin 7x}{\sin 3x}$$

$$\lim_{x \to 0} \frac{\sin 7x}{7x} \cdot \frac{7x}{1} \cdot \frac{3x}{\sin 3x} \cdot \frac{1}{3x}$$

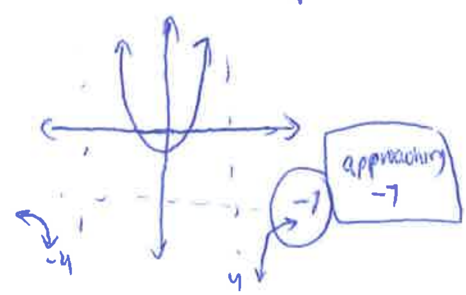
$$\lim_{x \to 0} \frac{\sin 7x}{\sin 3x} = \boxed{\frac{7}{3}}$$

#4 $\lim_{x \to \infty}$ HORIZONTAL ASYMPTOTE

$$\lim_{x \to \infty} \frac{5x - 7x^2}{x^2 - 16}$$

VA = ±4

HA = $\frac{-7}{1}$ = −7



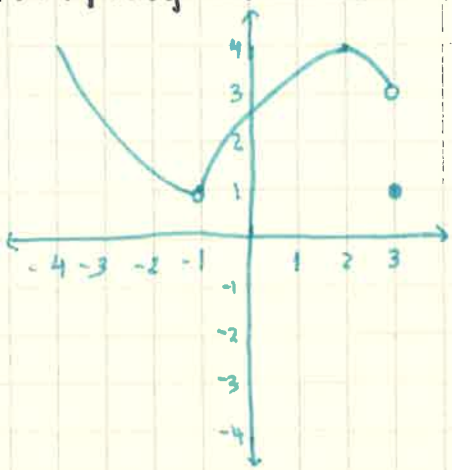
Limits Involving Horizontal Asymptote

1. $\lim\limits_{x\to\infty} \dfrac{7x-5x^2}{x^2-9}$

$\lim\limits_{x\to\infty} \dfrac{7x-5x^2}{1(x)^2-9}$ ← horizontal asymptote = $\boxed{-5}$

Identifying Limits w/ Graphs

2.



$\lim\limits_{x\to 3} f(x) =$ 3

$\lim\limits_{x\to -1^+} f(x) =$ 1

$f(3) =$ 1

Finding Derivatives

3. $f(x) = \sqrt{3x-1}$ ← finding the derivatives of the following using the definition $\left(\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}.\right)$

$$f'(x) = \lim_{h\to 0} \frac{\left[\sqrt{3(x+h)-1}-\sqrt{3x-1}\right]}{h} \cdot \frac{\left[\sqrt{3(x+h)+1}+\sqrt{3x-1}\right]}{\left[\sqrt{3(x+h)+1}+\sqrt{3x-1}\right]}$$

$$\lim_{h\to 0} \frac{3(x+h)-1-(3x-1)}{h\left[\sqrt{3(x+h)-1}+\sqrt{3x-1}\right]} = \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left[\sqrt{3(x+h)-1}+\sqrt{3x-1}\right]}$$

$$= \frac{3}{\sqrt{3x-1}+\sqrt{3x-1}} = \boxed{\frac{3}{2\sqrt{3x-1}}} \text{ OR } \boxed{\frac{3\sqrt{3x-1}}{6x-2}}$$

↑ substitute h with 0

CH. 18 REVIEW

RULES:

$f(x) = \frac{u}{v} \rightarrow f'(x) = \frac{u'v - uv'}{v^2}$

ex. find the derivative of $\frac{\tan 5x}{6x^4 - x^2}$

$u = \tan 5x \qquad v = 6x^4 - x^2$
$u' = \sec^2(5x)(5) \qquad v' = 24x^3 - 2x$

$\rightarrow \boxed{\frac{\sec^2(5x)(5)(6x^4 - x^2) - (\tan 5x)(24x^3 - 2x)}{(6x^4 - x^2)^2}}$

$f(x) = u \cdot v \rightarrow f'(x) = u'v + uv'$

ex. find the derivative of $e^{(x^2-5x)} \sec(2x)$

$u = e^{(x^2-5x)} \qquad v = \sec(2x)$
$u' = e^{(x^2-5x)}(2x-5) \qquad v' = \sec(2x)\tan(2x)(2)$

$\rightarrow \boxed{e^{(x^2-5x)}(2x-5)\sec 2x + e^{(x^2-5x)}\sec 2x \tan 2x \cdot 2}$

- $f(x) = \sin^6(x - 5x^3)$, $f'(x) = ?$

① $6\sin^5(x - 5x^3)$ [derivative of ~~cos~~ $\sin(x - 5x^3)$]
② $6\sin^5(x - 5x^3)\cos(x - 5x^3)$
③ $6\sin^5(x - 5x^3)\cos(x - 5x^3)(1 - 15x^2)$

- $f(x) = \log_5(7x^2 + \pi)$, $f'(x) = ?$

$f'(x) = \frac{1}{\ln 5(7x^2 + \pi)}(14x)$

Challenge Question

Find $\frac{dy}{dx}$ if $y = \sqrt{x}(2x - 8x^7)^3$

Ch 14 Review

1) Chain Rule

Find $\frac{dy}{dx}$ of $y = (x^2 - 2x)^4$

$f(x) = x^2 - 2x$ so $f'(x) = 2x - 2$

$\frac{dy}{dx} = [(x^2 - 2x)^4]'$

$\frac{dy}{dx} = 4(x^2 - 2x)^3 \cdot f'(x)$

$\boxed{\frac{dy}{dx} = 4(x^2 - 2x)^3 \cdot (2x - 2)}$

Example: Find $\frac{dy}{dx}$ if $y = \sqrt{x}\,(2x - 8x^7)^3$

Chain rule applies

$(x^{\frac{1}{2}})'(2x - 8x^7)^3 + x^{\frac{1}{2}}[(2x - 8x^7)^3]'$

$(\frac{1}{2})x^{-\frac{1}{2}}(2x - 8x^7)^3 + x^{\frac{1}{2}}(3)(2x - 8x^7)^2(2 - 56x^6)$

$\boxed{\frac{dy}{dx} = \frac{(2x - 8x^7)^3}{2\sqrt{x}} + 3\sqrt{x}\,(2x - 8x^7)^2(2 - 56x^6)}$

Implicit Differentiation (1st and Second Derivatives)

Group 3

Blake Harrison P2

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Implicit Differentiation is when the function is expressed in both y and x, rather than just x.

Instead of .... $y = x^2 + 4x$

Implicit... $y^2 + x^2 = 14$

When deriving y... ALWAYS MULTIPLY BY $\left(\frac{DY}{DX}\right)$ or (y').

Example:

Differentiate $x^2 + 4y^2 = 1$ to the first and second derivatives.

① $\boxed{y'}$

(power rule) → $\left(\frac{d}{dx}\right) x^2 + 4y^2 = 1\left(\frac{d}{dx}\right)$

$$2x + 8y(y') = 0$$
$$-2x \qquad\qquad -2x$$

(isolate y) →

(isolate y′) → $\frac{8y}{8y}(y') = \frac{-2x}{8y}$

(simplify for y′) → $\boxed{y' = \frac{-x}{4y}}$

② $\boxed{y''}$ Derive y′

(quotient rule) → $y' = \frac{-x}{4y}$

(derive x and y) → $\frac{(-x)'4y - (-x)(4y)'}{(4y)^2}$

$$\frac{(-1)4y - (-x)(4y)'}{(4y)^2}$$

(Simplify) → $\frac{-4y + 4x(y')}{16y^2}$

(substitute y′) → $y'' = \frac{-y + xy'}{4y^2}$

$$y'' = \frac{-y + x\left(\frac{-x}{4y}\right) \cdot 4y}{4y^2 \cdot 4y}$$

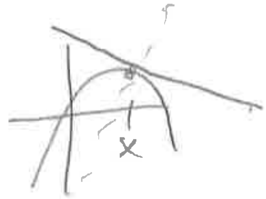
$$\frac{-4y^2 - x^2}{16y^3}$$

REMEMBER
$x^2 + 4y^2 = 1$
so
$-4y^2 - x^2 = -1$

$$y'' = \frac{-1}{16y^3}$$

Groups

Equations of tangents and normal



similar to point-slope form

equation of tangent: $y - f(c) = f'(c) \cdot (x - c)$ $\Rightarrow y - y_1 = a \cdot (x - x_1)$

coordinate is $c, f(c)$

normal: $y - f(c) = -\frac{1}{f'(c)} \cdot (x - c)$ $\Rightarrow$ slope is negative reciprocal

Example: Find the equation(s) of the tangent and normal to $f(x) = \sqrt{5x}$ at $x = 5$.

$f(x) = \sqrt{5} \cdot x^{\frac{1}{2}}$

1) $\frac{dy}{dx} = \sqrt{5} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}$

$= \frac{\sqrt{5}}{2} \cdot \frac{1}{\sqrt{x}}$

$= \frac{\cancel{\sqrt{5}}}{2} \cdot \frac{1}{\cancel{\sqrt{5}}}$

$\frac{dy}{dx} = \frac{1}{2}$

2) $f(5) = \sqrt{5 \cdot 5}$

$= 5$

3) tangent: $y - 5 = \frac{1}{2}(x - 5)$

normal: $y - 5 = -2(x - 5)$

☆ $\frac{dy}{dx}$ may include x and/or y variables, which then requires plugging in coordinate (x, y) to get the derivative

Group 3

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Derivatives of Inverse Trig Functions

~rules

$\sin^{-1}(x) = \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

$\cos^{-1}(x) = \frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$

$\tan^{-1}(x) = \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

$\cot^{-1}(x) = \frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}$

$\sec^{-1}(x) = \frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x|\sqrt{x^2-1}}$

$\csc^{-1}(x) = \frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{|x|\sqrt{x^2-1}}$

Example:

$$\frac{d}{dx}(3(\cos^{-1}(2x^3-7)) = 3\cdot\left(\frac{-1}{\sqrt{1-(2x^3-7)^2}}\right)\cdot 6x^2 = 18x^2\cdot\frac{-1}{\sqrt{1-(2x^3-7)^2}}$$

$$\frac{d}{dx}(x\cdot\sin^{-1}x) = \sin^{-1}(x) + \frac{1}{\sqrt{1-x^2}}\cdot x = \sin^{-1}(x) + \frac{x}{\sqrt{1-x^2}}$$

Logarithmic Differentiation

~rules

$\ln ab = \ln(a) + \ln(b)$ $\quad \boxed{\ln(x)}' = \frac{1}{x}$ $\quad \ln\frac{a}{b} = \ln(a) - \ln(b)$ $\quad \ln(b^a) = a\ln(b)$

ex. $y = \frac{x^3\cdot\sqrt[4]{x^5-5x}}{\cos(x)}$

ex. $\ln 3b = \ln 3 + \ln b$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\ln\left(\frac{x^3\cdot\sqrt[4]{x^5-5x}}{\cos(x)}\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}\left[\ln x^3 + \ln(x^5-5x)^{\frac{1}{4}} - \ln(\cos(x))\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}\left[3\ln x + \frac{1}{4}\ln(x^5-5x) - \ln(\cos(x))\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 3\cdot\frac{1}{x} + \left(\frac{1}{4}\cdot\frac{1}{x^5-5x}\cdot(5x^4-5)\right) - \left(\frac{1}{\cos(x)}\cdot -\sin(x)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{x} + \frac{5x^4-5}{4x^5-20x} + \tan(x) \rightarrow \frac{dy}{dx} = \left(\frac{3}{x} + \frac{5x^4-5}{4x^5-20x} + \tan(x)\right)y$$

$$\frac{dy}{dx} = \left(\frac{x^3\sqrt[4]{x^5-5x}}{\cos(x)}\right)\cdot\left(\frac{3}{x} + \frac{5x^4-5}{4x^5-20x} + \tan(x)\right)$$

Chapter 20: MVT, Rolle's Theorem, Continuity, Differentiability, Kinematics

Main Concepts:

Differentiability: Function f is differentiable if and only if it is differentiable at every value of x in its domain

f is differentiable at $x = a$ if: right-hand derivative $\lim_{x \to a^+} f'(x)$ and the left-hand derivative $\lim_{x \to a^-} f'(x)$ both exist and equal to each other

Continuity: If a function f is differentiable at $x = c$, then f is continuous at $x = c$

Rolle's Theorem: If $f(a) = f(b)$, then there exists a value $c \in]a,b[$ such that $f'(c) = 0$

MVT: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Kinematics: Distance: $s(t)$: The position of the object on the line is a function of time, t

Velocity: $v(t) = \dfrac{ds}{dt} = \lim_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$: rate of change of $s(t)$ with respect to time

Average Velocity: $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$; speed: $|v(t)|$

Acceleration: $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = \lim_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$: rate of change of $v(t)$ with respect to time

Instantaneous Rate of Change: $\left.\dfrac{df}{dx}\right|_{x=a}$: The instantaneous rate of change of $f(x)$ with respect to x at $x = a$ is the value of the derivative of $f(x)$ at $x = a$

Examples:

Show that $f(x) = \begin{cases} \cos x, & x \geq 0 \\ x^2+3x+1, & x<0 \end{cases}$ is continuous but not differentiable at $x=0$

Work: $\lim_{x \to 0^+} \cos x = \lim_{x \to 0^-} x^2+3x+1 \longrightarrow \cos(0) = 1$

$1 = 1$

$\therefore f(x)$ continuous at $x=0$

$f(0)$ exists

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0^+} (\cos x)' = \lim_{x \to 0^-} (x^2+3x+1)'$

$= \lim_{x \to 0^+} -\sin x = \lim_{x \to 0^-} 2x+3$

$0 \neq 3$ $\therefore f(x)$ not differentiable at $x=0$

Find all numbers within $[2,5]$ that satisfy MVT for $f(x)=x^2$

slope: $\frac{f(2)-f(5)}{2-5} = 7$

$f'(x) = 2x$

$2x = 7$ $x = 3.5$

- A ball thrown upward at $t = 0$ has height $h(t) = -4.9t^2 + 24t + 7$ meters after t seconds.

a. Find the ball's velocity and acceleration at time t.

$v(t) = -9.8t + 24t$ $\quad$ $a(t) = -9.8$

b. When does the ball reach it's maximum height and what is it?

$-9.8t + 24 = 0 \quad t = 24/9.8 = 2.45$ seconds

$h(2.54) = -4.9(2.45)^2 + 24(2.45) + 7 = 36.4$ meters $\quad$ t $\;$ 2.45 $\;$ + $\;$ − $\quad$ 2.45 is a local max

c. When does the ball hit the ground and at what velocity?

$0 = -4.9t^2 + 24t + 7$ $\quad$ $t = 5.17$ second $\quad v(5.17) = -9.8(5.17) + 24 = -26.7$ meters / second

d What is the total distance traveled by the ball?

$|36.4 - 7| + |0 - 36.4| = 65.8$ m

A particle moves in a straight line so that it's position ~~after t seconds~~ is $s(t) = t^3 - 4t^2 + 10t + 4$

a Write expressions for velocity and acceleration.

$v(t) = 3t^2 - 8t + 10$ $\quad$ and $\quad$ $a(t) = 6t - 8$

b What intervals is the particle's speed increasing?

Speed Increasing or Decreasing

- In a kinematic problem, to find out whether the speed is increasing or decreasing in an interval you find the roots of the acceleration and velocity equations and then make sign diagrams for both. When the velocity and acceleration have the same sign the speed is increasing.