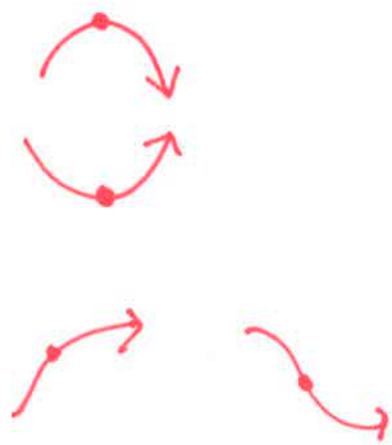
Curve Analysis

STATIONARY POINTS $(f'(a)=0)$

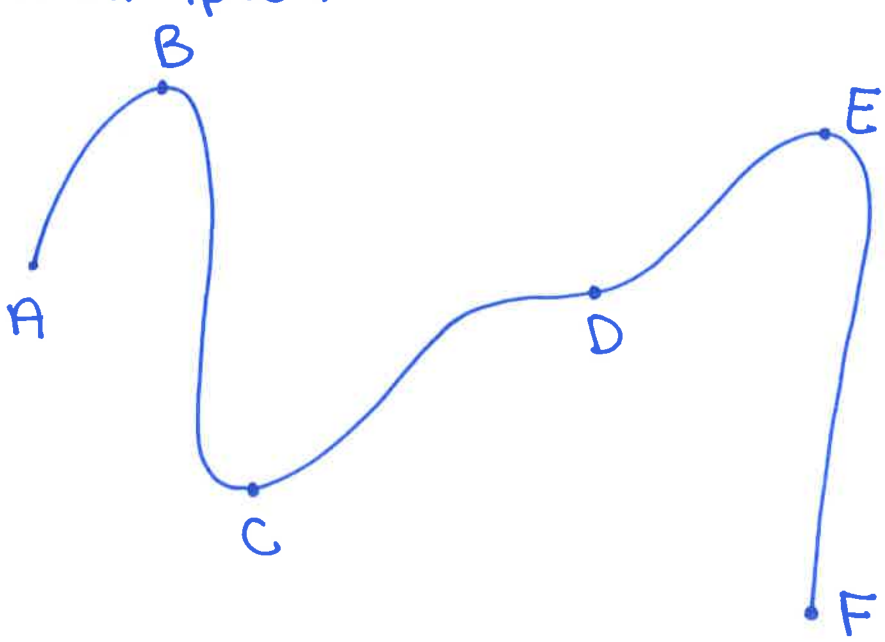
local max: $\oplus \rightarrow \ominus$

local min: $\ominus \rightarrow \oplus$

Stationary inflection: $\ominus \rightarrow \ominus$, $\oplus \rightarrow \oplus$



example 1:

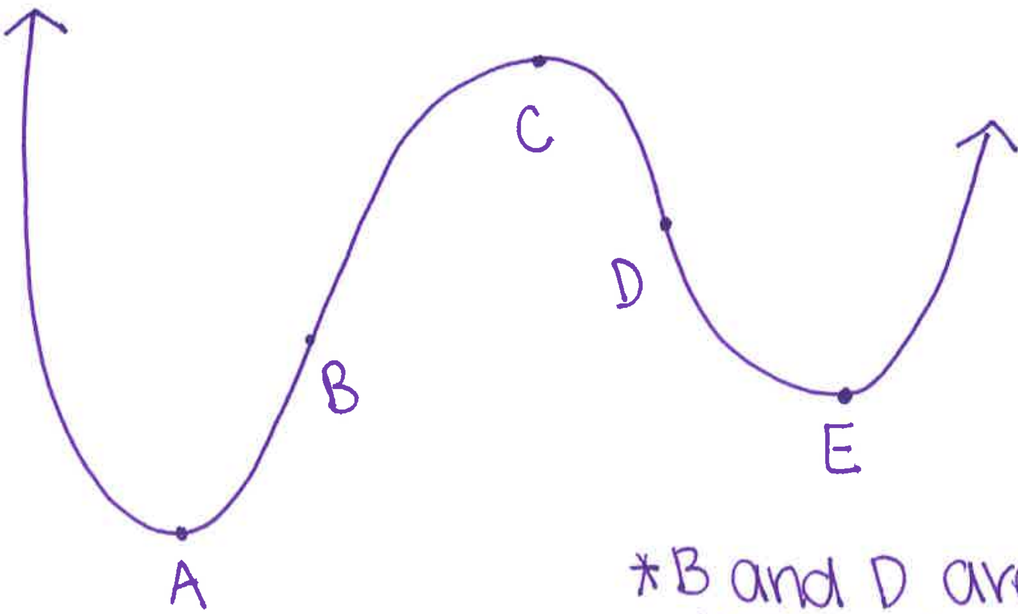


Local max:
B, E

Local min:
C

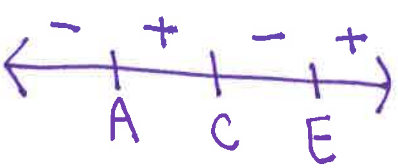
Stationary Inflection:
D

Function h(x):



*B and D are points of inflection.

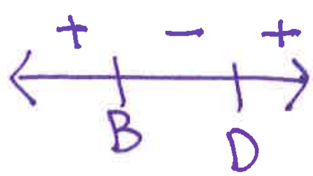
$h'(x)$



increasing:
$(A, C) \cup (E, \infty)$

decreasing:
$(-\infty, A) \cup (C, E)$

$h''(x)$



concave up:
$(-\infty, B) \cup (D, \infty)$

concave down:
(B, D)

Examples are missing.

* you need to find "Examples" and study.

③

optimization

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FINDING THE MAXIMUM OR MINIMUM BASED ON A SET OF CONSTRAINTS.

#1 MAKE DIAGRAM WITH APPROPRIATE LABELS

#2 CREATE A FUNCTION TO FIND THE DESIRED VARIABLE (MIGHT NEED MORE THAN ONE)

#3 TAKE DERIVATIVE OF THE FUNCTION

#4 SET IT EQUAL TO ZERO AND FIND THE VARIABLE

#5 USE THE VALUE OF VARIABLE TO DETERMINE THE FINAL ANSWER

#6 CONFIRM THE ANSWER YOU FOUND IS A MAX OR MIN BY USING 2nd DERIVITIVE TEST / 1st DERIVITIVE SIGN DIAGRAM (IF APPLICABLE)

related rate

FINDING A RATE AT WHICH A QUANTITY CHANGES IN RESPECT TO TIME.

#1 DRAW A DIAGRAM AND STATE GIVEN INFO USING PROPER RATE OF CHANGE NOTATION

#2 SET UP PROPER MATHEMATICAL EQUATION(S) USING VARIABLES.

#3 DIFFERENTIATE EQUATIONS WITH RESPECT TO TIME (t).

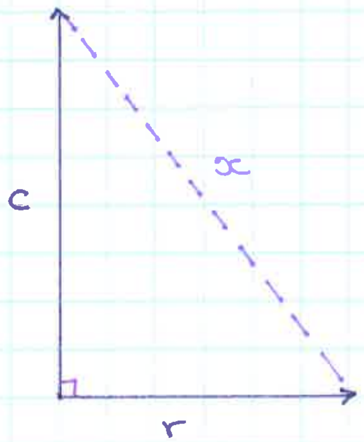
#4 CALCULATE THE RATE OF CHANGE USING GIVEN INFO. GIVE EXACT ANSWER IN 3 SIG FIGS.

4

Example Questions

#1 A car and truck start in the same place. The car heads north at 60 mph, the truck goes east at 40 mph. How fast is the distance between the two increasing at 2 hours?

1



$$\frac{dc}{dt} = 60\text{mph}$$

$$\frac{dr}{dt} = 40 \text{ mph}$$

$$\frac{dx}{dt} = ?$$

$c = 60 \times 2 = 120$

$r = 40 \times 2 = 80$

$x = \sqrt{120^2 + 80^2} \approx 144.22$

2 $c^2 + r^2 = x^2$

3 $\cancel{2}c \cdot \frac{dc}{dt} + \cancel{2}r \cdot \frac{dr}{dt} = \cancel{2}x \cdot \frac{dx}{dt}$

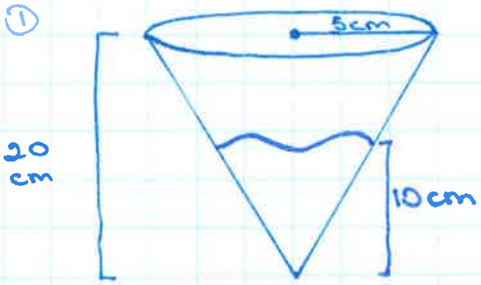
4 $120(60) + 80(40) = 144.22 \frac{dx}{dt}$

$10400 = 144.22 \frac{dx}{dt}$

$$\boxed{\frac{dx}{dt} = 72.1 \text{ mph}}$$

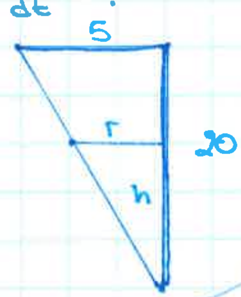
#2 A cone with a height of 20cm and a radius of 5cm is filled with water. It leaks at a rate of 12 cm³/s. How fast is the water level falling when the water is 10 cm deep?

1



$$\frac{dV}{dt} = 12\text{cm}^3/\text{s}$$

$$\frac{dh}{dt} = ?$$



$$\frac{20}{5} = \frac{h}{r}$$

$20r = 5h$

$r = \frac{1}{4}h$

2 $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$

$V = \frac{1}{3 \times 16}\pi h^3$

$V = \frac{1}{48}\pi h^3$

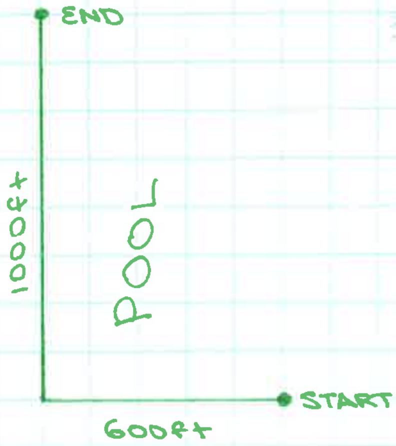
3 $\frac{dV}{dt} = \frac{1}{16}\pi h^2 \cdot \frac{dh}{dt}$

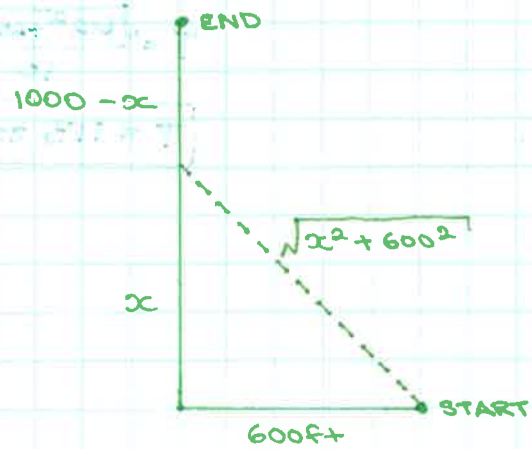
4 $12 = \frac{1}{16}\pi(100) \cdot \frac{dh}{dt}$

$$\boxed{\frac{dh}{dt} = 0.611 \text{cm/s}}$$

#3 Yolanda is in a race that requires swimming and running. She figures her speed is 14 ft/s for running and 10 ft/s for swimming (faster than Phelps). Yolanda wants to decide her route to take the fastest time to complete the race.

a) Sketch a possible route of the least time diagram.





b) Find the equation for the time, T in seconds, taken crossing the pool and running on the ground to finish the race.

$$T = \frac{\sqrt{x^2+600^2}}{10} + \frac{1000-x}{14}$$

c) Find $\frac{dT}{dx}$.

$$\frac{dT}{dx} = \frac{1}{10}\left(\frac{1}{\cancel{2}}\right)(\cancel{2}x)(x^2+600^2)^{-\frac{1}{2}} - \frac{1}{14}$$

$$\frac{dT}{dx} = \frac{x}{10\sqrt{x^2+600^2}} - \frac{1}{14}$$

d) $\frac{dT}{dx} = 0$. Solve for x.

$$0 = \frac{x}{10\sqrt{x^2+600^2}} - \frac{1}{14}$$

$$\frac{1}{14} = \frac{x}{10\sqrt{x^2+600^2}}$$

$$10\sqrt{x^2+600^2} = 14x$$

$$\sqrt{x^2+600^2} = \frac{7}{5}x$$

$$x^2+600^2 = \frac{49}{25}x^2$$

$$-\frac{24}{25}x^2 = -600^2$$

$$\sqrt{x^2} = \sqrt{\frac{600^2 \cdot 25}{24}}$$

$$x = 612$$

Optimisation

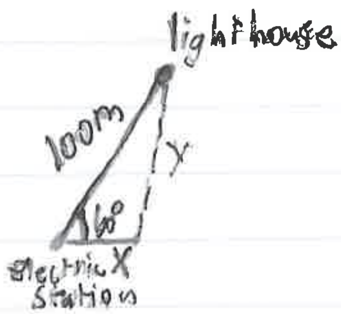
- Optimisation is the process by which one finds the optimum solution, or the maximum or minimum of a function.
- Problem Solving Method:
    1. Draw a large, clear diagram of the situation.
    2. Construct a formula with a single variable to be optimised. Also include any domain restrictions there are on said variable.
    3. Find the first derivative and the values that make the derivative zero.
    4. Prove using the second derivative test that you have a maximum or a minimum.

Related Rates

- Related rates involve a differential equation to describe the motion of an object at any instant.
- Related Rates Solving Method:
    1. Draw a large, clear diagram(s) of the situation.
    2. Write down the information, label the diagram(s), and make sure you distinguish between the variables and the constants.
    3. Write an equation connecting the variables. You will often need to use Pythagoras' theorem, similar triangles, triangle trignometry, sine and cosine rules, etc.
    4. Differentiate the equation with respect to t to obtain a differential equation.
    5. Solve.

7

An electricity station is on the edge of a straight coastline. A lighthouse is located 100 meters from the station and the angle between the line joining both and the coast is 60°. A cable is going from the lighthouse straight to the coast then x distance to the station. The cable cost 0.7 m per dollar ~~and \$10~~ on land and \$10 per meter on the sea. Find ~~the minimum cost of x.~~ X when the cost is minimum.



$$Cost = 10y + \frac{x}{0.7}$$

$$y = \sqrt{100^2 + x^2 - 2 \cdot 100 \cdot x \cdot \cos(60°)}$$

$$y = \sqrt{x^2 - 100x + 10000}$$

equation for cost → $Cost = 10\sqrt{x^2 - 100x + 10000} + \frac{10}{7}x$

$$Cost' = 5(x^2 - 100x + 10000)^{-\frac{3}{2}} \cdot (2x - 100) + \frac{10}{7}$$

$$Cost' = \frac{10x - 500}{\sqrt{(x^2 - 100x + 10000)^3}} + \frac{10}{7}$$

GFC: $0 = \frac{10x - 500}{\sqrt{(x^2 - 100x + 10000)^3}} + \frac{10}{7}$

$$\boxed{x \approx 10.7}$$

Check if minimum

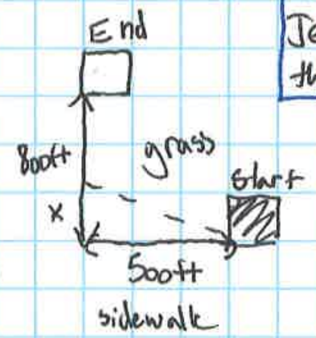
$$Cost'' = \frac{10(x^2 - 100x + 10000)^{\frac{3}{2}} + (15x - 750)(x^2 - 100x + 10000) \cdot (2x - 100)}{(x^2 - 100x + 10000)^3}$$

∴ What we found was a minimum

8

Jennifer makes daily trips from the Start to the end building. Her speed on the Sidewalk is 5ft/s and her speed is 4ft/s on grass. Jennifer wants to take the fastest route from the start to the end. Sketch the possible route of the least time diagram.

a)



Find an equation for the Time T in seconds taken crossing grass and sidewalk the rest of the way.

b) $T = \frac{1}{4}\sqrt{x^2+500^2} + \frac{1}{5}(800-x)$

Find $\frac{dT}{dx}$

c) $\frac{dT}{dx} = \frac{1}{4}\cdot\frac{1}{2}(x^2+500^2)^{-\frac{1}{2}}\cdot 2x - \frac{1}{5}$

$\frac{dT}{dx} = \frac{x}{4}(x^2+500^2)^{-\frac{1}{2}} - \frac{1}{5}$

Solve $\frac{dT}{dx} = 0$

d)

$0 = \frac{1}{8}\cdot\frac{1}{\sqrt{x^2+500^2}}(2x) - \frac{1}{5}$

$\frac{1}{5} = \frac{x}{4}\cdot\frac{1}{\sqrt{x^2+500^2}}$

$\frac{\sqrt{x^2+500^2}}{5} = \frac{x}{4}$

$5x = 4\sqrt{x^2+500^2}$

$\left(\frac{5x}{4}\right)^2 = x^2+500^2$

$\frac{25}{16}x^2 - x^2 = 500^2$

$\frac{9}{16}x^2 = 500^2$

$x^2 = 44444.444$

$\boxed{x = 667\text{ ft}}$

Related Rate

9

1. A tank with water is in the shape of an inverted cone 20 ft high with a circular base on the top whose radius is 5 ft. Water is running out of the bottom of the tank at ~~the~~ 18 ft^3/s. How fast is the water level falling when the water is 8 ft deep.

$h = 8$

$\frac{dh}{dt} = ?$

$\frac{dv}{dt} = -18 \frac{ft^3}{s}$

$\frac{h_2\ 20}{(r_2)\ 5} = \frac{8\ (h_1)}{r\ (r_1)}$

$20 r = 40$

$r = 2$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dt} = \frac{1}{3}\pi \cdot 2 \cdot r \cdot \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt}$

$\frac{20}{5} = \frac{h}{r} \rightarrow r = \frac{1}{4}h$

$-18 = \frac{1}{3}\pi \cdot 2 \cdot 2 \cdot 8 \cdot \frac{\frac{dh}{dt}}{4} + (2)^2 \cdot \frac{dh}{dt}$

$-18 = \frac{dh}{dt}\left(\frac{1}{3} \cdot \frac{1}{4} \cdot 32 \cdot \pi + 4\right)$

$\boxed{\frac{dh}{dt} = -1.43\ ft/s}$

Chapter 14: Vectors

A scalar is a value of magnitude such as speed
A vector is a value of magnitude in a direction such as velocity

Vectors can exist in 2D or 3D, being represented as $\begin{pmatrix} i \\ j \end{pmatrix}$ or $\begin{pmatrix} i \\ j \\ k \end{pmatrix}$ respectively. This is component form. All vectors have a unit vector, where you divide the values by the magnitude to get a vector with magnitude one $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \to \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$
The unit-vector form is made when the vector is written $\vec{v} = i + j \, (+k)$, with coefficients being the corresponding values in component form.

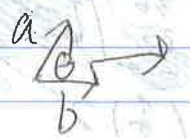
Vectors can be added: $\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} + \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} = \begin{pmatrix} i_1 + i_2 \\ j_1 + j_2 \end{pmatrix}$; subtracted: $\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} - \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} = \begin{pmatrix} i_1 - i_2 \\ j_1 - j_2 \end{pmatrix}$; but can be multiplied in two ways. the scalar or dot product is the sum of the products of corresponding components: $\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \cdot \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} = (i_1 * i_2) + (j_1 * j_2)$; the vector or cross product only works on 3D vectors:

$$\begin{pmatrix} i_1 \\ j_1 \\ k_1 \end{pmatrix} \times \begin{pmatrix} i_2 \\ j_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} j_1 k_2 - k_1 j_2 \\ k_1 i_2 - i_1 k_2 \\ i_1 j_2 - j_1 i_2 \end{pmatrix}$$

Vectors can be multiplied/divided by scalars, and would then be parallel

$$x \begin{pmatrix} i_1 \\ j_1 \\ k_1 \end{pmatrix} \parallel \begin{pmatrix} i_1 \\ j_1 \\ k_1 \end{pmatrix},$$ the vectors have the same slopes

The dot product is used to find the angle between two vectors



$\cos\theta = \dfrac{a \cdot b}{|a||b|}$, which is useful (when $a \cdot b = 0$, $\theta = 90°$)

notation of magnitude

The cross product is a vector that is perpendicular to both vectors by the right-hand rule. Also its magnitude is 2(Area of triangle with the vectors)
(cross products can be found as 3x3 matrix's determinant also by the way)

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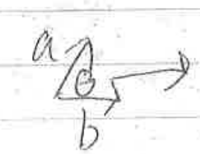

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magnitude = 62.1

area of A = $\frac{\text{magnitude}}{2}$

$\frac{62.1}{2} = \boxed{31.05 = \text{area}}$

$\frac{12+20+56}{\sqrt{79}\sqrt{105}} = \frac{88}{\sqrt{79}\sqrt{105}}$

$\arccos \frac{88}{\sqrt{79}\sqrt{105}} \boxed{\approx 84.8^\circ}$