Mathematical Induction: Proof of a conjecture.
Day Three: Trigonometry and Complex Number

Math Induction proof process for a given conjecture (statement):
Step 1: Show that the statement is true for an initial case, n=1.
Step 2: Assume that the statement is true for n=k where $k \in Z^+$.
Step 3: Prove that the statement is true for n=k+1.
Step 4: ∴ The statement is true for $n \in Z^+$

Problem 1)

Prove using Mathematical induction that $P(n)$ is true for all $n \in Z^+$.

$$P(n): \frac{1}{2} + \cos x + \cos 2x \ldots\ldots\ldots \cos nx = \frac{1}{2}\frac{\sin(\frac{2n+1}{2}x)}{\sin(\frac{1}{2}x)}$$

1) when $n=1$

$L.H.S = \frac{1}{2} + \cos x$, $RHS = \frac{1}{2}\frac{\sin(\frac{3}{2}x)}{\sin(\frac{1}{2}x)} = \frac{\sin(\frac{x}{2}) + \sin(\frac{3}{2}x) - \sin(\frac{x}{2})}{2\sin(\frac{x}{2})}$

$= \frac{1}{2} + \left[\frac{\sin(\frac{3}{2}x) - \sin(\frac{x}{2})}{2\sin(\frac{x}{2})}\right] = \frac{1}{2} + \cos x$

Notes: $\sin(\frac{3}{2}x) - \sin(\frac{x}{2}) = 2\cos\left(\frac{\frac{3}{2}x + \frac{x}{2}}{2}\right)\sin\left(\frac{\frac{3}{2}x - \frac{x}{2}}{2}\right) \Rightarrow \cos x = \frac{\sin(\frac{3}{2}x) - \sin(\frac{x}{2})}{2\sin(\frac{x}{2})}$

Identity (Difference as a product)
$\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$

$p(n)$ is true for $n=1$

2) when $n=k$, Assume $p(n)$ is true.

$\frac{1}{2} + \cos x + \cos 2x \ldots \cos kx = \frac{1}{2}\frac{\sin(\frac{2k+1}{2}x)}{\sin(\frac{1}{2}x)}$

3) If $n=k+1$,

$\frac{1}{2} + \cos x + \cos 2x \cdots \cos kx + \cos(k+1)x = \frac{1}{2}\frac{\sin(\frac{2k+1}{2}x)}{\sin(\frac{1}{2}x)} + \cos(k+1)x$

$= \frac{\sin\left(\frac{2k+1}{2}x\right)}{2\sin(\frac{1}{2}x)} + \frac{2\cos((k+1)x)\sin(\frac{1}{2}x)}{2\sin(\frac{1}{2}x)} = \frac{\sin\left(\frac{2k+1}{2}x\right)}{2\sin(\frac{1}{2}x)} + \frac{\left(\sin\frac{2k+3}{2}x - \sin\frac{2k+1}{2}x\right)}{2\sin(\frac{1}{2}x)}$

$= \frac{\sin\left(\frac{2k+3}{2}x\right)}{2\sin\frac{x}{2}} = \frac{\sin\left(\frac{2(k+1)+1}{2}x\right)}{2\sin(\frac{x}{2})}$

4) ∴ $p(n)$ is true for $n \in Z^+$

Problem 2)

Use the principle of mathematical induction to prove that $\arg(z^n) = n\arg(z)$ for all $n \in Z^+$.

1) $n=1 \Rightarrow \arg(z) = \arg(z)$
$p(n)$ is true for $n=1$

$p(n)$: proposition that $\arg(z^n) = n\arg(z)$

2) When $n=k$, Assume $\arg(z^k) = k\arg(z)$ is true.

3) If $n=k+1$, $\arg(z^{k+1}) = \arg(z^k \cdot z) = \arg(z^k) + \arg(z)$
$= k\arg(z) + \arg(z)$
$= (k+1)\arg(z)$.

4) $\therefore$ $\arg(z^n) = n\arg(z)$ is true for $n \in Z^+$

IB Questions) Do on separate paper

1

Use mathematical induction to prove that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, $n \in \mathbb{Z}^+$.

2

Prove $(i+1)^n = (\sqrt{2})^n(\cos(\frac{n\pi}{4}) + i\sin(\frac{n\pi}{4}))$ for $n \in Z^+$ by mathematical induction.

a)

3.

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$.

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$.

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$.

#1. $p(n)$ proposition that $z^n = r^n(\cos n\theta + i\sin n\theta)$

$z^n = (r\,\text{cis}\,\theta)^n$

1) when $n=1$

$r(\cos\theta + i\sin\theta)' = r'(\cos\theta + i\sin\theta)$

$p(n)$ is true for $n=1$.

2) When $n=k$,

Assume $(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta)$ is true.

3) If $n=k+1$,

$(r(\cos\theta + i\sin\theta))^{k+1} = r\cdot \boxed{r^k(\cos\theta + i\sin\theta)^k}(\cos\theta + i\sin\theta)$

$= r\cdot r^k(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta]$

$= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$

$= r^{k+1}[\cos\theta(k\theta + \theta) + i\sin\theta(k\theta + \theta)]$

$= r^{k+1}[\cos\theta(\theta(k+1)) + i\sin(\theta(k+1))]$

4) $\therefore$ $p(n)$ is true for $n \in \mathbb{Z}^+$.

#2. $p(n)$: proposition that $(1+i)^n = (\sqrt{2})^n\cos(\frac{n\pi}{4}) + i\sin(\frac{n\pi}{4})$

$1+i = \sqrt{2}\,\text{Cis}(\frac{\pi}{4})$.

1) when $n=1$ $\Rightarrow \sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})) = \sqrt{2}'(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$

$p(n)$ is true for $n=1$.

2) when $n=k$, Assume $(\sqrt{2}\,\text{Cis}(\frac{\pi}{4}))^k = (\sqrt{2})^k(\cos(\frac{k\pi}{4}) + i\sin(\frac{k\pi}{4}))$ is true.

3) When $n=k+1$

$[\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^{k+1} = \sqrt{2}\boxed{\sqrt{2}^k(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^k}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$= \sqrt{2}\cdot\sqrt{2}^k(\cos\frac{\pi k}{4} + i\sin\frac{\pi k}{4})(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

→

$$= \sqrt{2}^{k+1}\Big[\cos\tfrac{\pi k}{4}\cdot\cos\tfrac{\pi}{4} + i\cos\tfrac{\pi k}{4}\sin\tfrac{\pi}{4} + i\sin\tfrac{\pi k}{4}\cos\tfrac{\pi}{4} - \sin\tfrac{\pi k}{4}\cdot\sin\tfrac{\pi}{4}\Big]$$

$$= \sqrt{2}^{k+1}\Big(\cos\left(\tfrac{\pi k}{4}+\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi k}{4}+\tfrac{\pi}{4}\right)\Big]$$

$$= \sqrt{2}^{k+1}\Big(\cos\tfrac{\pi}{4}(k+1) + i\sin\tfrac{\pi}{4}(k+1)\Big)$$

4) $\therefore$ $p(n)$ is true for $n \in \mathbb{Z}^+$

#3. See attached.

13. (a) $\sin\ (2n+1)x\ \cos x - \cos\ (2n+1)x\ \sin x = \sin\ (2n+1)x - x$ ***M1A1***

$= \sin 2nx$ ***AG***

[2 marks]

(b) if $n = 1$ ***M1***

LHS $= \cos x$

RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x \cos x}{2\sin x} = \cos x$ ***M1***

so LHS $=$ RHS and the statement is true for $n = 1$ ***R1***

assume true for $n = k$ ***M1***

> **Note:** Only award ***M1*** if the word **true** appears.
> Do **not** award ***M1*** for 'let $n = k$' only.
> Subsequent marks are independent of this ***M1***.

so $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \dfrac{\sin 2kx}{2\sin x}$

if $n = k+1$ then

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$ ***M1***

$= \dfrac{\sin 2kx}{2\sin x} + \cos(2k+1)x$ ***A1***

$= \dfrac{\sin 2kx + 2\cos\ (2k+1)x\ \sin x}{2\sin x}$ ***M1***

$= \dfrac{\sin\ (2k+1)x\ \cos x - \cos\ (2k+1)x\ \sin x + 2\cos\ (2k+1)x\ \sin x}{2\sin x}$ ***M1***

Sin (2kx+x − x)
= Sin [(2k+1)x − x]
= Sin [(2k+1)x] cosx − cos[(2k+1)x] Sinx

compound Angle identity

$= \dfrac{\sin\ (2k+1)x\ \cos x + \cos\ (2k+1)x\ \sin x}{2\sin x}$ ***A1***

compound Angle identity

$= \dfrac{\sin\ (2k+2)x}{2\sin x}$ ***M1***

$= \dfrac{\sin\ 2(k+1)x}{2\sin x}$ ***A1***

so if true for $n = k$, then also true for $n = k+1$

as true for $n = 1$ then true for all $n \in \mathbb{Z}^+$ ***R1***

> **Note:** Final ***R1*** is independent of previous work.

[12 marks]

continued …

(12)

Question 13 continued

(c) $\dfrac{\sin 4x}{2\sin x} = \dfrac{1}{2}$ ***M1A1***

$\sin 4x = \sin x$

$4x = x \Rightarrow x = 0$ but this is impossible

$4x = \pi - x \Rightarrow x = \dfrac{\pi}{5}$ ***A1***

$4x = 2\pi + x \Rightarrow x = \dfrac{2\pi}{3}$ ***A1***

$4x = 3\pi - x \Rightarrow x = \dfrac{3\pi}{5}$ ***A1***

for not including any answers outside the domain ***R1***

Note: Award the first ***M1A1*** for correctly obtaining $8\cos^3 x - 4\cos x - 1 = 0$ or equivalent and subsequent marks as appropriate including the answers $\arccos\left(-\dfrac{1}{2}, \dfrac{1 \pm \sqrt{5}}{4}\right)$.

[6 marks]

Total [20 marks]