Matteo S., Jason K., Victoria Z., & Anna Z.

CHAPTER 17: limits

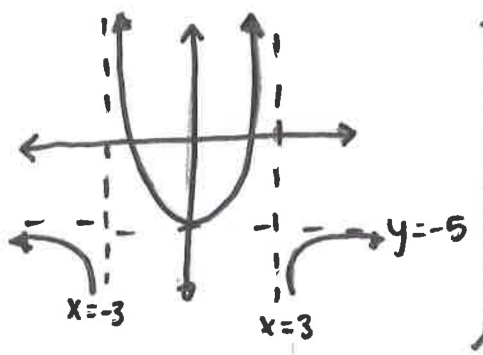
LIMIT: the value that a function approaches (as the input approaches some value)

LIMITS AT INFINITY:

- try to solve algebraically if possible.
- if not — graph it.

ex: $\lim_{x \to \infty} \frac{7x - 5x^2}{x^2 - 9}$

$\lim_{x \to \infty} \frac{-x(5x-7)}{(x+3)(x-3)}$

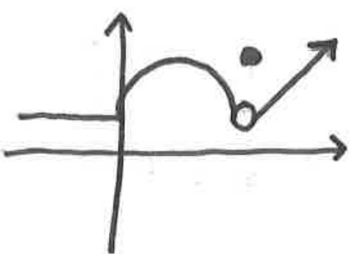


$\boxed{\lim_{x \to \infty} -5}$

CONTINUITY:

• vs. o

$f(3)$ vs. $\lim_{x \to 3} f(x)$



THAT ONE PROBLEM WE ALL HATE: (first principle differentiation)

Find the slope of the tangent line to $f(x) = -3x^2 + 2x$ at $x = 3$ using the derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\lim_{h \to 0} \frac{-3(x+h)^2 + 2(x+h) - (-3x^2 + 2x)}{h}$

$\lim_{h \to 0} \frac{-3x^2 - 6xh - 3h^2 + 2x + 2h + 3x^2 - 2x}{h}$

$\lim_{h \to 0} \frac{-6xh - 3h^2 + 2h}{h} = \lim_{h \to 0} -6x - 3h + 2 \rightarrow h = 0 \rightarrow -3x + 2$

$f'(x) = -6x + 2$

$f'(3) = -18 + 2 = -16$

$f'(3) = -6(3) + 2$

$= -18 + 2$

$= \boxed{-16}$

TRIG

②

$$\lim_{x \to 0} \frac{1-\cos^2 x}{6x^2}$$

$$\lim_{x \to 0} \frac{\sin^2 x}{6x^2}$$

$$\lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{6}$$

PROPERTY:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\boxed{\lim_{x \to 0} \frac{1}{6}}$$

Questions

PIECE-WISE

1. Find the constants A and B so that the given function will be continuous for all x. ③

$$f(x) = \begin{cases} Ax+B & \text{if } x<5 \\ -3 & \text{if } x=5 \\ x^2+Ax-B & \text{if } x>5 \end{cases}$$

answer:

$Ax+B=-3$

$x^2+Ax-B=-3$

$Ax+B = x^2+Ax-B$

$-Ax+B \qquad -Ax+B$

$2B = x^2 \rightarrow x=5$

$2B=5^2$

$2B=25$

$B=25/2$

$Ax+B=-3$

$A(5)+25/2=-3$

$A(5)=-31/2$

$A=-31/10$

check work:

$Ax+B=-3$

$(-31/10)(5)+(25/2)=-3$

$-31/2+25/2=-3$

$-6/2=-3$ ✓

$x^2+Ax-B=-3$

$(5)^2+(-31/10)(5)-(25/2)=-3$

$25-31/2-25/2=-3$

$19/2-25/2=-3$

$-6/2=-3$ ✓

FIRST-PRINCIPLE

2. Find the slope of the tangent line at $x=2$ for $f(x)=4x^2-3$ using the first principle

$$\lim_{h\to0} \frac{[4(x+h)^2-3]-[4(x)^2-3]}{h}$$

$$\lim_{h\to0} \frac{4(x^2+2xh+h^2)\cancel{-3}-4x^2\cancel{+3}}{h}$$

$$\lim_{h\to0} \frac{\cancel{4x^2}+8xh+4h^2\cancel{-4x^2}}{h}$$

$$\lim_{h\to0} 8x+4h = 8x$$

$$f'(2)=\lim_{h\to0} 8(2)+4(0)$$

$$\lim_{h\to0} 16$$

$$\boxed{\text{slope} = 16 \text{ at } x=2}$$

3. $f(x) = \begin{cases} x+2, & x<1 \\ a(x-3)^2, & x \geq 1 \end{cases}$

PIECE WISE

4

Find value of a for which $f(x)$ is continuous at $x=1$

Step 1: Find each one-sided limit in terms of a

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1^-} (x+2) = \lim_{x\to 1^+} (a(x-3)^2)$$

Step 2: assume the limit exists and solve for a

$$1+2 = a(1-3)^2$$

$$3 = a(4)$$

$$\boxed{a = \frac{3}{4}}$$

4.

TRIG

Chapter 18: RULES of DIFFERENTIATION

6

POWER RULE: simple differentiation $f(x)=x^n$ $f'(x)=n(x^{n-1})$

CHAIN RULE: when a function is inside another function $y=(f(x))^n$

$$\frac{dy}{dx}=n(f(x))^{n-1}\cdot f'(x)$$

EXAMPLE QUESTION:

Find the derivative: $f(x)=(2-3x)^2+2x\sqrt{x}-29x^4$

1) simplify equation: $f(x)=(2-3x)^2+2x^{\frac{3}{2}}-29x^4$

2) use power and chain rule: $f'(x)=[\underbrace{2(2-3x)\cdot(-3)}_{-6(2-3x)}]+[\frac{3}{2}\cdot 2\cdot x^{\frac{1}{2}}]-[29\cdot 4\cdot x^3]$

$$f'(x)=(-12+18x)+(3\sqrt{x})-(116x^3)$$

$$\boxed{f'(x)=-12+18x+3\sqrt{x}-116x^3}$$

Rachel, Steven, Alexis, Junior

6

THE PRODUCT RULE:

When differentiating the product of two expressions,

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

THE QUOTIENT RULE:

When differentiating the quotient of two expressions

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

*Don't forget to apply POWER rule and CHAIN rule

EX: Differentiate $f(x) = x^2(x-5)^4 + \frac{x-4}{2x+1}$

$$\frac{dy}{dx} = 2x(x-5)^4 + x^2(4)(x-5)^3 + \frac{(1)(2x+1)-(x-4)(2)}{(2x+1)^2}$$

$$\frac{dy}{dx} = 2x(x-5)^4 + 4x^2(x-5)^3 + \frac{9}{(2x+1)^2}$$

$$\frac{dy}{dx} = 2x(x-5)^3((x-5)+2x) + \frac{9}{(2x+1)^2}$$

$$\frac{dy}{dx} = 2x(x-5)^3(3x-5) + \frac{9}{(2x+1)^2}$$

IMPLICIT DIFFERENTIATION

- taking the derivative of y ($\frac{dy}{dx}$) in implicit equations
- you can take the derivative of y in any equation the same way as x, but you have to multiply the derivative of y ($\frac{dy}{dx}$)(y')

ex: xy^2

↳ $y^2 + x2y(y')$

DERIVATIVES OF EXPONENTIAL QUESTIONS/FUNCTIONS

- $\frac{d}{dx}(b^x) = b^x \ln b$
- $\frac{d}{dx}(e^x) = e^x \ln e = e^x$
- $\frac{d}{dx}\left(e^{f(x)}\right) = \frac{de^f}{df}\frac{df}{dx} = e^f \frac{df}{dx}$

Example Question

Find $\frac{dy}{dx}$ for the equation $3x^2e^y + y^6 = 20x$

① $6x^2e^y + 3x^2e^y(y') + 6y^5(y') = 20$

② $3x^2e^y(y') + 6y^5(y') = 20 - 6x^2e^y$

③ $y'(3x^2e^y + 6y^5) = 20 - 6x^2e^y$

④ $\boxed{y' = \frac{20 - 6x^2e^y}{3x^2e^y + 6y^5}}$

Derivatives of Trig ~~&~~ Functions

The curve C is given by $y = \frac{x\cos x}{x+\cos x}$, for $x \geq 0$

a) Show that $\frac{dy}{dx} = \frac{\cos^2 x - x^2 \sin x}{(x+\cos x)^2}$, $x \geq 0$

$u = x\cos x$
$u' = \cos x - \sin x$
$v = x + \cos x$
$v' = 1 - \sin x$

$$\frac{(\cos x - \sin x)(x+\cos x) - (x\cos x)(1-\sin x)}{(x+\cos x)^2}$$

$$= \frac{x\cos x - x^2\sin x - x\sin x\cos x - x\cos x + x\sin x\cos x}{(x+\cos x)^2}$$

$$= \boxed{\frac{\cos^2 x - x^2\sin x}{(x+\cos x)^2}}$$

Derivative of Implicit Differentiation with $\tan^{-1} u$ Example.

A curve has equation $\arctan x^2 + \arctan y^2 = \frac{\pi}{4}$

(a) Find $\frac{dy}{dx}$ in terms of x and y

$$\frac{2x}{1+x^4} + \frac{2y}{1+y^4} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\left(\frac{2x}{1+x^4}\right)\left(\frac{1+y^4}{2y}\right) = \frac{-x(1+y^4)}{y(1+x^4)}$$

$x = \frac{1}{\sqrt{2}} \Rightarrow \cancel{x^2 = \frac{1}{2}} \arctan\left(\frac{1}{2}\right) + \arctan y^2 = \frac{\pi}{4}$

$\cancel{\tan(x^2 + y^2)} + \cancel{\frac{\tan x^2 + \tan y^2}{1}} = \frac{\frac{1}{2} + y^2}{1 \cdot y^2} = \tan\frac{\pi}{4} = 1$

b) $\cancel{\frac{1}{2} - y^2 = 1 + \frac{1}{2}y^2}$

$\cancel{\frac{3}{2}y^2 = \frac{1}{2}}$ $\quad y = \pm\frac{1}{\sqrt{3}}$ $\quad y = \frac{1}{\sqrt{3}}$ $(y > 0)$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{\sqrt{2}}, \cancel{y=\frac{1}{\sqrt{3}}}} = \frac{-\frac{1}{\sqrt{2}}\left(1+\frac{1}{9}\right)}{\frac{1}{\sqrt{3}}\left(1+\frac{1}{4}\right)} = \frac{-\sqrt{3}}{\sqrt{2}} =$$

Equations of tangent + normal lines

- Finding the equation of a tangent line to graph $f(x)$ at given point $x=c$

$$y - f(c) = f'(c)(x-c)$$

- Finding the equation of a normal line to the graph $f(x)$ at given point $x=c$

$$y - f(c) = -\frac{1}{f'(c)}(x-c)$$

example: Find the equation(s) of the tangent line and normal line to the graph of $f(x) = \sqrt{5x}$ at $x=5$

step 1: find $\frac{dy}{dx}$

$\rightarrow f(x) = \sqrt{5x} = (5x)^{\frac{1}{2}} = \sqrt{5} \cdot x^{\frac{1}{2}}$

$\frac{dy}{dx} = \sqrt{5} \cdot \frac{1}{2}x^{-1/2}$

$= \frac{\sqrt{5}}{2\sqrt{x}}$

step 2: find the y-coordinate in $f'(x)$

$f'(x) = \frac{\sqrt{5}}{2\sqrt{x}}$

$f'(5) = \frac{\sqrt{5}}{2\sqrt{5}}$

$f'(5) = 1/2$

step 3: find the y-coordinate in $f(x)$

$f(x) = \sqrt{5x}$

$f(5) = \sqrt{5 \cdot 5}$

$f(5) = 5$

$\therefore$ tangent line: $(5,5)$, $m = \frac{1}{2}$

$\Leftrightarrow \boxed{(y-5) = \frac{1}{2}(x-5)}$

$\therefore$ normal line: $(5,5)$, $m = -2$ (reciprical)

$\Leftrightarrow \boxed{(y-5) = -2(x-5)}$

Implicit Differentiation

↳ higher order derivatives

↳ if y is in the function x, then

$$y' = \frac{d}{dx}(y) = \frac{dy}{dx} \qquad y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \qquad y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Example: If $x^2 + y^2 = 4$, show that $\frac{d^2y}{dx^2} = \frac{x^2+y^2}{y^3}$

Step 1: Find $\frac{dy}{dx}$

$y = x^2 + y^2 = 4$

$\frac{dy}{dx} = \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(4)$

$= 2x + 2y\left(\frac{dy}{dx}\right) = 0$

$\frac{2y\left(\frac{dy}{dx}\right)}{2y} = \frac{-2x}{2y}$

$\frac{dy}{dx} = -\frac{x}{y}$

$\frac{dy}{dx} = \frac{-2x}{2y}$

Step 2: take the derivative of $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-x}{y}$

$= \frac{(-x)'(y) - (-x)(y)'}{y^2}$

$= \frac{-y + x\left(\frac{dy}{dx}\right)}{y^2}$

$= \frac{-y + x\left(\frac{-x}{y}\right)(y)}{y^2 \quad (y)}$

$= \frac{-y^2 - x^2}{y^3} \cdot -1$

$= \boxed{\frac{y^2 + x^2}{y^3}}$

$y = g(x)$

$y = f(x)$

$(x+6)^2 \qquad g(x) = x + 6$

$g'(x)\frac{dg}{dx}$

$2(x+6)(1/1)$

$\frac{d}{dx}(y) \qquad \frac{dy}{dy} \qquad \frac{dy}{}$

$\frac{d}{dx}(f(g(x)))$

Derivatives of Inverse Trig

$$\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}} \; ; \quad \frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2} \; ; \quad \frac{d(\cot^{-1}x)}{dx} = -\frac{1}{1+x^2}$$

$$\frac{d(\csc^{-1}x)}{dx} = -\frac{1}{|x|\sqrt{x^2-1}} \; ; \quad \frac{d(\sec^{-1}x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$$

Example:

1. $y = \cos^{-1}(x^2)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^4}} \times 2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

LOGARITHMIC *functions*

Find $\frac{dy}{dx}$ by logarithmic differentiation for $y = \dfrac{e^{2x}\sqrt{x-3x^2}}{(\sin x)(5x^2+x)^7}$

$$\frac{dy}{dx} \cdot \frac{1}{y} = \ln e^{2x} + \ln (x-3^x)^{\frac{1}{2}} - \ln(\sin x) - \ln(5x^2+x)^7$$

$$= y\left[2x \ln e + \frac{1}{2} \ln (x-3^x)^{-\frac{1}{2}} - \ln(\sin x) - 7 \ln(5x^2+x)\right] = \frac{dy}{dx}$$

$$= y\left(2 + \frac{1-6x}{2(x-3x^2)} - \frac{\cos x}{\sin x} - \frac{7(10x+1)}{(5x^2+x)}\right] = \frac{dy}{dx}$$

$$\Rightarrow y\left[2 + \frac{1-6x}{2(x-3x^2)} - \cot x - \frac{7(10x+1)}{(5x^2+x)}\right] = \frac{dy}{dx}$$

RULES:

$$\frac{d}{dx}(\log_b x) = \frac{1}{(\ln b)\cdot x}$$

$(x+1)^3 = 3(x+1)^2(1)$

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)} \cdot u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Continuity

—Major Concepts—

$f(x)$ is continuous at $x=a$ if:

1) $f(a)$ is defined

2) $\lim_{x\to a} f(x)$ exists

3) $f(a) = \lim_{x\to a} f(x)$

—Example—

Prove that $f(x) = \begin{cases} x, & x>1 \\ x^2, & x=1 \\ x^3, & x<1 \end{cases}$ is continuous at $x=1$.

$f(1) = (1)^2$

$f(1) = 1$

$\lim_{x\to 1^-} f(x) = (1)^3$

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$

$\therefore \lim_{x\to 1} f(x)$ exists

$\lim_{x\to 1} f(x) = f(1) = 1$

$\therefore f(x)$ is continuous at $x=1$

Shreyas Madhvaraju

Rolle's Theorem:

- Suppose a function $f: D \to R$ is continous on the closed interval $[a, b]$ and differentiable at the open interval $]a, b[$ and if $\boxed{f(a) = f(b) = 0}$, then there exists a value such that $c \in]a, b[$ such that $\boxed{f'(c) = 0}$

Mean Value Theorem (MVT):

- Suppose a function $f: D \to R$ is continuous on the closed interval $[a, b]$, and differentiable at the open interval $]a, b[$ then $\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$ for some number $c \in]a, b[$

Both Rolle's Theorem and MVT have the same theory, and follow different formulas.

eg. Show that function $f(x) = x^2 + 2x - 3$ satisfies the MVT on the closed interval $[1, 2]$ and find the number c between $(1, 2)$ that satisfies MVT

ans.) $a = 1 \quad b = 2$

$f(1) = -1$

$f'(x) = 2x + 2$

$f(2) = 5$

$\boxed{f'(c) = 2c + 2}$

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$2c + 2 = \frac{6}{1}$$

$$2c = 4$$

$$\boxed{c = 2}$$

Arvind Chithambaram

Differentiability

A function $f: D \to R$ is differentiable at $x = a$ $a \in D$ if

1. f is continuous at $x = a$

2. Suppose f is a real function with domain D containing an open interval about $x = a$. f is differentiable at $x = a$ if $f'(a) = \lim\limits_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ and $f(a) = \lim\limits_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ both exist and are equal.

Prove that $\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$

$$\lim_{h \to 0} \frac{(\cos h - 1)}{h} \cdot \frac{(\cos h + 1)}{(\cos h + 1)} = \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$$

$$\lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{-\sin h}{(\cos h + 1)} = 1 \cdot \frac{-0}{(1+1)} = \boxed{0}$$

Example

Prove that $f(x) = \begin{cases} \sin x & x \geq 0 \\ x^2 + 5x & x < 0 \end{cases}$ is continuous but not differentiable at $x = 0$

Continuity

$f(0) = \sin 0 = 0$

$\lim\limits_{x \to 0^+} x = 0$ $\lim\limits_{x \to 0^-} x^2 + 5x = 0$

$\therefore$ fx is continuous at 0

Differentiability

$\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} \cos 0 = \boxed{1}$ $1 \neq 5$

$f(x)$ is not differentiable

$\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} 2(0) + 5 = \boxed{5}$

Sean Steck

Kinematics

Summary: kinematics reffers to the tracking of motion, specifically position or distance, velocity and acceleration.

Distance: $S(t)$: The position of an object on a line at time t

Velocity: $V(t) = \frac{ds}{dt}$: The rate of change of distance with respect to time

Acceleration: $a(t) = \frac{dV}{dt} = \frac{d^2s}{dt^2}$: The rate of change of velocity in respect to time

Example Problem:

A moving object is moving at an equation that can be modled by the equation $s(t) = 3t^3 - 50t^2 + 362t + 4$

1. What is the initial position, velocity and acceleration?

$S(t) = 3t^3 - 50t^2 + 362t + 4$
$\frac{ds}{dt} = 9t^2 - 100t + 362 = V(t)$
$\frac{dV}{dt} = 18t - 100 = a(t)$

$S(0) = 3(0)^3 - 50(0)^2 + 362(0) + 4 = \boxed{4}$
$V(0) = 9(0)^2 - 100(0) + 362 = \boxed{362}$
$a(t) = 18(0) - 100 = \boxed{-100}$

2. When does the object change directions?

$V(t) = 0 \Rightarrow 0 = 9t^2 - 100t + 362 \Rightarrow x \approx -2.88$
$x \approx 14.0$

time $\neq 0$ $\therefore$ $\boxed{t \approx 14 \text{ seconds}}$

Speed is increasing when velocity and acceleration are the same sign and decreasing when they are different signs.