Curve Analysis

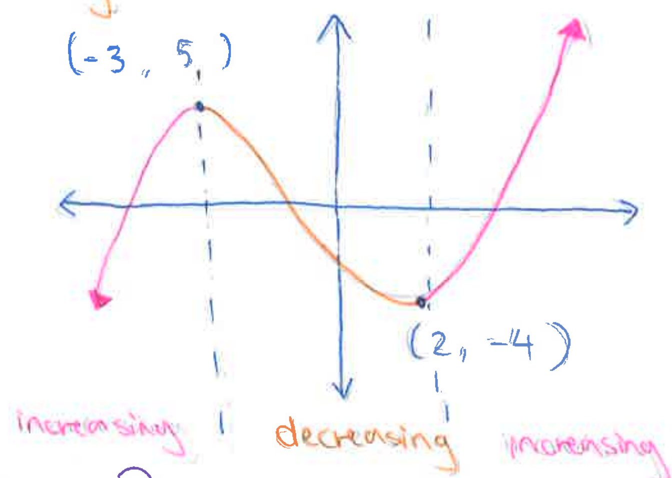
19B Increasing and Decreasing Functions

The function $f(x)$ is . . .

Increasing when the $f'(x) > 0$

&

Decreasing when the $f'(x) < 0$



19C Stationary Points

Stationary Point where $f'(a) = 0$	Sign diagram of $f'(x)$ near $x = a$	Shape of curve near $x = a$
local maximum	$+$ a $-$	
local minimum	$-$ a $+$	
stationary inflection	$+ \rightarrow +$ or $- \rightarrow -$	or

* all zeros of 1st derivative are stationary points

* inflection points are zeros of 2nd derivative

* the overlap between zeros of 1st derivative & zeros of second derivatives is called stationary inflection

2

1.) Find the greatest (local max) and least value (local min) of $y = x^3 - 9x^2 + 7$ on interval $(1, 4)$

$y' = 3x^2 - 18x$

$0 = 3x(x-6)$

$x = 0,\ x = 6$

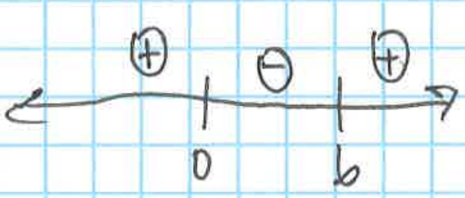
$y(0) = 7$

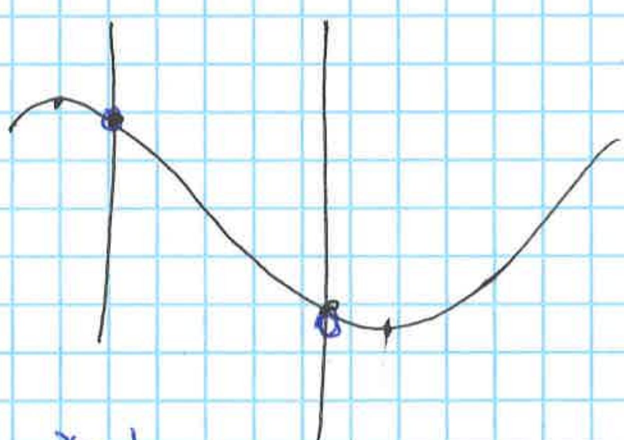
$y(6) = -101$

$y(1) = -1$

$y(4) = -41$

$$\boxed{\begin{array}{l} \text{Absolute max: } (1, -1) \\ \text{Absolute min: } (4, -41) \end{array}}$$





2) The distance of a ball t seconds after it is launched is

$$s(t) = \frac{2}{3}t^3 - 24t^2 + t - 42 \text{ miles.} \quad t \geq 0$$

a) Find equations of the velocity and acceleration of the ball.

Solution: $v(t) = \frac{ds}{dt}$

$$\boxed{v(t) = 2t^2 - 48t + 1 \ \frac{\text{miles}}{s}}$$

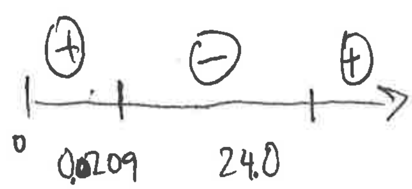
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$\boxed{a(t) = 4t - 48 \ \frac{\text{miles}}{s^2}}$$

b) Figure out when the ball's speed is increasing and decreasing. Use sign diagrams. $0 \leq t \leq 26$

Solution: $v(t) = 0$

$$t = \frac{48 \pm \sqrt{2296}}{4}$$



3 sig figs

$$t = \frac{24 + \sqrt{574}}{2} \approx 23.97 \approx 24.0$$

$$t = \frac{24 - \sqrt{574}}{2} \approx 0.0209$$

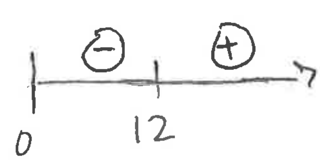
Speed Increasing: $(0.0209, 12) \cup (24, 26)$

$\boxed{\text{Speed decreasing: } (0, 0.0209) \cup (12, 24)}$

$a(t) = 0$

$4t = 48$

$t = 12$



continued on back

3) $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$

a) Determine the stationary points.

$f'(x) = 12x^3 - 48x^2 + 48x$

$f'(x) = 12x(x^2 - 4x + 4)$

$f'(x) = 12x(x-2)^2$

-4
-2 -2
4

stationary points at $x = 0$ $x = 2$

⊖ ⊕ ⊕

0 2

$x = -1$ $x = 1$ $x = 3$

b) determine inflection points

$f''(x) = 36x^2 - 96x + 48$

-8
-6 -2
12

$f''(x) = 12(3x^2 - 8x + 4)$

$f''(x) = 12(3x^2 - 6x - 2x + 4)$

$f''(x) = 12(3x(x-2) - 2(x-2))$

$f''(x) = 12(3x-2)(x-2)$

inflection points: $x = 2/3$ $x = 2$

c) what is the stationary point of inflection?

$x = 2$

Group 6 Declan K. Rithvik T. Conner L. Amanda Z. 6

Related Rate Steps

1) Draw a diagram and state the given information using proper rate of change notation

2) Set up proper mathematical equation(s) using variables

3) Differentiate the equation(s) with respect to time

4) Calculate the rate of change using the given information
   Give an exact answer or round to three sig figs.

Optimization

1. Write equation of what you're trying to solve

2. Find ~~[illegible]~~ derivative

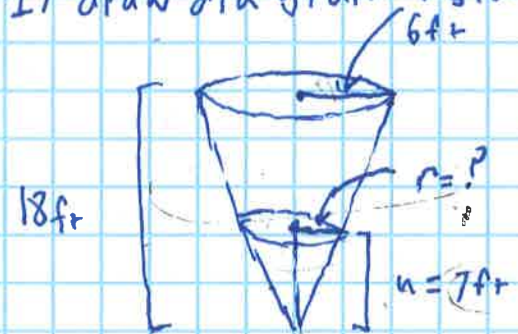
3. Solve for x and plug back into equation

4. find 2nd derivative and prove maximum/minimum

7

A tank with water is in the shape of an inverted cone with height of 18 feet. The circular base ontop has a radius of six feet. water is added to the tank at a rate of $9 ft^3/s$. how fast is the water level rissing if the water is 7ft deep

1) draw diagram & state information



$r = ?$

$h = 7ft$

$\frac{dv}{dt} = 9 ft^3/s$

$\frac{dh}{dt} = ?$

2) Set up proper equations

$$V = \frac{1}{3} \cdot \pi r^2 \cdot h$$

$$\frac{6}{18} = \frac{r}{h} \Rightarrow \frac{1}{3} = \frac{r}{h} \Rightarrow 3r = h$$

3) differentiate with respect to time

$$\frac{dv}{dt} = \frac{1}{3} \cdot \pi \left(r^2 \cdot h' + r^{2'} \cdot h \right)$$

$$\frac{dv}{dt} = \frac{1}{3} \cdot \pi \left(r^2 \cdot \frac{dh}{dt} + 2r \cdot \frac{dr}{dt} \cdot h \right)$$

$$3 \frac{dr}{dt} = \frac{dh}{dt}$$

4) Calculate rate of change. Give exact answer & round to three sig figs.

$$\frac{dr}{dt} = \frac{1}{3} \cdot \frac{dh}{dt}$$

$3r = h$

$3r = 7$

$r = \frac{7}{3}$

$$9 = \frac{1}{3} \cdot \pi \left(\left(\frac{7}{3}\right)^2 \cdot \frac{dh}{dt} + 2\left(\frac{7}{3}\right) \cdot \frac{1}{3} \cdot \frac{dh}{dt} \cdot 7 \right)$$

$$9 = \frac{1}{3} \cdot \pi \left(\frac{49}{9} \cdot \frac{dh}{dt} + \frac{98}{9} \cdot \frac{dh}{dt} \right)$$

$$\frac{27}{\pi} = \frac{dh}{dt} \left(\frac{49}{9} + \frac{98}{9} \right)$$

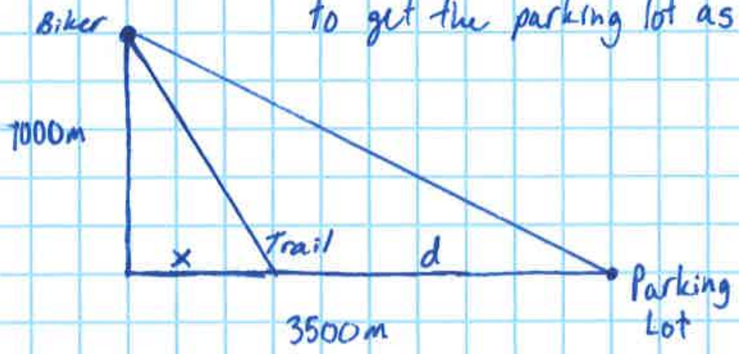
$$\frac{27}{\pi} = \frac{49}{3} \cdot \frac{dh}{dt}$$

$$\frac{81}{49\pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = .526 ft/s$$

$\boxed{\text{Water level is rising at a rate of } .526 ft \cdot s^{-1}}$

A biker is 1000 m from a straight trail. He needs to reach a parking lot 3500m downtrail from the closest point to him on the trail. If he bikes at 6 m/s when he is off the trail and at 10 m/s on the trail, how far from the parking lot should he get on the trail to get the parking lot as soon as possible?



$$\frac{dt}{dx} = 0$$

$$t = \frac{1000^2 + x^2}{6} + \frac{3500-x}{10} = \frac{1}{6}(1000^2+x^2)^{\frac{1}{2}} + \frac{1}{10}(3500-x)$$

$$\frac{dt}{dx} = \left(\frac{1}{6}\right)\left(\frac{1}{\cancel{2}}\right)(\cancel{2}x)(1000^2+x^2)^{-\frac{1}{2}} - \frac{1}{10}$$

$$0 = \frac{x}{6\sqrt{1000^2+x^2}} - \frac{1}{10}$$

$$+\frac{1}{10} \qquad +\frac{1}{10}$$

$$(10 \cdot 6\sqrt{1000^2+x^2})\left(\frac{1}{10}\right) = \left(\frac{x}{6\sqrt{1000^2+x^2}}\right)(10 \cdot 6\sqrt{1000^2+x^2})$$

$$(6\sqrt{1000^2+x^2})^2 = (10x)^2$$

$$\overset{9}{\cancel{36}}(1000^2+x^2) = \overset{25}{\cancel{100}}x^2$$

$$9(1000000+x^2) = 25x^2$$

$$9000000 + 9x^2 = 25x^2$$

$$-9x^2 \quad -9x^2$$

$$\frac{9000000}{16} = \frac{16x^2}{16}$$

$$\sqrt{562500} = \sqrt{x^2}$$

$$\boxed{750 = x}$$

$3500 - x =$ distance from parking lot

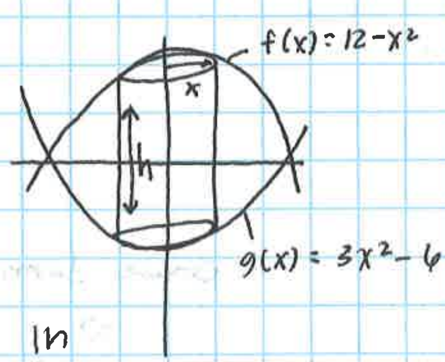
$3500 - x = d$

$3500 - 750 = d$

$$\boxed{2750 = d}$$

9

A cylinder is inscribed between the graphs of $f(x) = 12-x^2$ and $g(x) = 3x^2-6$ so that each side is parallel to the x or y-axis. h and x are labeled in the diagram. Units are in cm.



a) Show that $h = 18-4x^2$

$f(x) - g(x) = (12-x^2) - (3x^2-6)$

$= \boxed{-4x^2+18}$

b) Write down the volume of the cylinder in terms of x

$\boxed{V = \pi x^2 \cdot (18-4x^2)}$

c) Find the maximum volume of the cylinder

$V' = 2\pi x\,(18-4x^2) + \pi x^2(-8x) = 0$

$= 36\pi x - 8\pi x^3 + (-8\pi x^3)$

$V' = 36\pi x - 16\pi x^3$?

$(9-4x^2)(4\pi x) = 0$

$-4x^2(4\pi x) = -9$

$-16\pi x^3 = -9$

$x^3 = \frac{9}{16\pi}$

$x = \sqrt[3]{9/16\pi}$

$x = \pm\sqrt{\frac{9}{4}}$

$= \pm\frac{3}{2}$

Plug into V' equation

$\boxed{V' = 47.9 \text{ cm}^3}$

Back

Group 7: Optimization + Related Rates

10

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Optimization problem solving strategy:

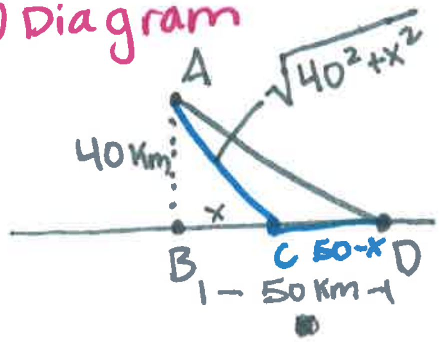
1. Draw a diagram of a given situation with appropriate notation.
2. Construct a formula with the variable to be optimized as the subject. (Remember to write in one variable using the given restriction).
3. Find the first derivative and solve for x to make the first derivative zero.
4. Confirm if the solution is maximum or minimum and revisit if the solution is reasonable.

Related Rates problem solving strategy:

1. Draw a large, clear diagram of the situation.
2. Write down the information, label the diagram, and make a distinction between variables and constants.
3. Write an equation connecting the variables.
4. Differentiate the equation with respect to t to obtain a differential equation.
5. Solve for the particular case which is at some instant in time.

A dune buggy is on the desert at point A located 40 Km from point B, which lies on a long straight road. The driver can travel at 45 Km/hr on the desert and 75 Km/hr on the road. The driver will win a prize if she arrives at the finish line at point D, 50 Km from B, in 84 min or less. What route should she travel to minimize the time of travel? Does she win the prize? 11

① Diagram



② Formula

$$t = \frac{\text{distance}}{\text{speed}}$$

$$t = \frac{\sqrt{40^2+x^2}}{45} + \frac{(50-x)}{75} = \frac{1}{45}(40^2+x^2)^{\frac{1}{2}} + \frac{1}{75}(50-x)$$

③ $\frac{dt}{dx} = 0$. Solve for x

$$\frac{dt}{dx} = \left(\frac{1}{45}\right)\left(\cancel{\tfrac{1}{2}}\right)\left(\cancel{2}x\right)(40^2+x^2)^{-\frac{1}{2}} + \left(\frac{1}{75}\right)(-1) = 0$$

$$\frac{dt}{dx} = \frac{x}{45\sqrt{40^2+x^2}} - \frac{1}{75} = 0$$

$$\frac{x}{45\sqrt{40^2+x^2}} = \frac{1}{75}$$

$$75x = 45\sqrt{40^2+x^2}$$

$$(5x)^2 = (3\sqrt{40^2+x^2})^2$$

$$25x^2 = 9(40^2+x^2)$$

$$25x^2 = 9(40^2) + 9x^2$$

$$16x^2 = 9(40^2)$$

$$x = \sqrt{\frac{9(40^2)}{16}} = \underline{30\text{ Km}}$$

④ Check

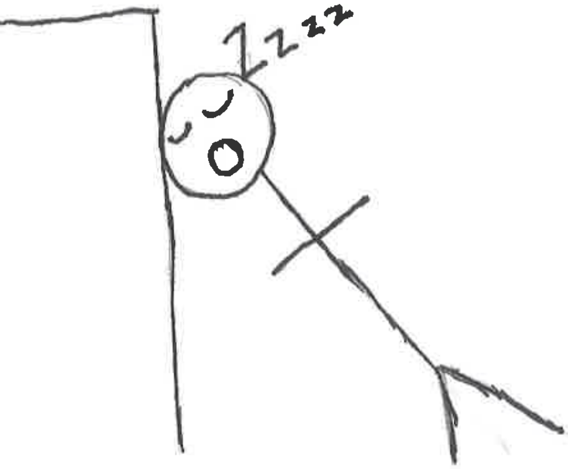
$$t = \left(\left(\frac{\sqrt{40^2+30^2}}{45}\right) + \left(\frac{50-30}{75}\right)\right) \cdot 60$$

$\approx \boxed{82.7 \text{ minutes}}$

yes, She does win the prize

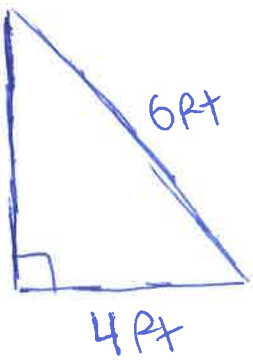
$\sqrt{40^2+30^2} =$ 50 Km through desert

$50 - 30 =$ 20 Km on road



Bill, who is 6ft tall, fell asleep resting against a vertical wall. Bill's feet are moving away from the wall at a constant speed of 0.5 ft/s. Calculate the speed of descent of the top of Bill when Bill's feet are 4ft away from the wall.

12



$$6^2 = x^2 + y^2$$

$$0 = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$

$$6^2 = y^2 + 4^2$$

$$36 = y^2 + 16$$

$$\sqrt{20} = y$$

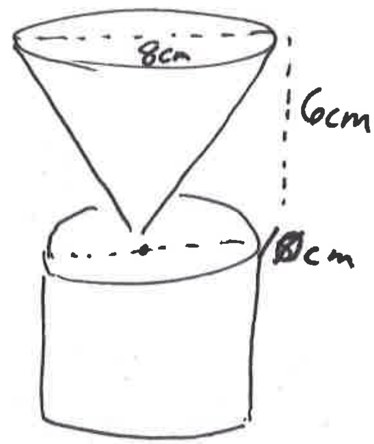
$$0 = 2(4)(0.5) + 2(\sqrt{20})\left(\frac{dy}{dt}\right)$$

$$0 = 4 + 2\sqrt{20}\left(\frac{dy}{dt}\right)$$

$$\boxed{-0.447 \text{ ft/s}}$$

Related Rates

$$V = \frac{1}{3}\pi r^2 h$$



The trick for related rates is to write one dimension/variable in terms of another.

$$d = \frac{8}{6}h$$

$$r = \frac{4}{6}h$$

$$\frac{dh}{dt} = 2\,cm\cdot s^{-1}$$

$$h = 3$$

* We should be given one of the rates dh dr or dV. This is the variable we will write the final equation in terms of.

$$V = \frac{1}{3}\pi\left(\frac{4}{6}\right)^2 \overbrace{\underbrace{(h)^2}_{3cm}\underbrace{h}_{3cm}}^{h^3}$$

$$\frac{dV}{dt} = \frac{3}{3}\pi\left(\frac{4}{6}\right)^2 \underbrace{h^2}_{3cm}\underbrace{\frac{dh}{dt}}_{2\,cm\,s^{-1}}$$

$$\frac{dV}{dt} = 2\pi\left(\frac{4}{6}\right)^2 \underbrace{h^2}_{3cm}$$

$$\frac{dV}{dt} = 2\pi\left(\frac{4}{6}\right)^2 3^2$$

$$\frac{dV}{dt} = 8\pi$$

$$V = \pi r^2 h$$

$$V = \pi(25)h$$

$$\underbrace{\frac{dV}{dt}}_{8\pi\,\frac{cm^3}{s}} = 25\pi\frac{dh}{dt}$$

$$\frac{\left(\frac{dV}{dt}\right)_{8\pi\,cm^3\cdot s^{-1}}}{25\pi} = \frac{dh}{dt}$$

$$\frac{8\cancel{\pi}}{25\cancel{\pi}} = \frac{dh}{dt}$$

Overall points:

Eliminate unknown values (variables) by writing them in terms of known values.

Take the derivative of the re-written equation

vectors

- a vector is a quantity with both magnitude and direction.
  ↳ represented by a directed line segment

→ component form: $\begin{pmatrix} a \\ b \end{pmatrix}$

→ unit vector form: $ai + bj + ck$

↳ a unit vector has a length of one unit.

→ dot product: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$; $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$$\boxed{v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3}$$

- if $v \cdot w > \emptyset$, θ is $\emptyset° < \theta < 90°$
- if $v \cdot w = \emptyset$, $\theta = 90°$
- if $v \cdot w < \emptyset$, θ is $90° < \theta < 180°$

→ cross product: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$; $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

OR

*use determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) \ldots$$
$$\ldots + k(a_1 b_2 - a_2 b_1)$$

↳ $\vec{a} \times \vec{a} = \emptyset$ for all $\vec{a}$

↳ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ for $\vec{a}$ and $\vec{b}$

↳ scalar triple product: $a \cdot (b \times c)$

↳ Can be used to find perpendicular vectors in relation to other vectors

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Vector Problems

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Vishnuth
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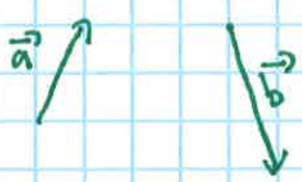
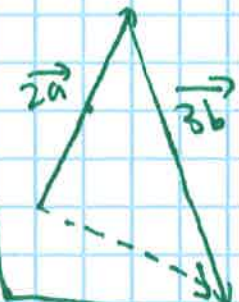
1.



Draw $\vec{a} + \vec{b}$

Draw $2\vec{a} + 3\vec{b}$



2. $\vec{a} = \begin{pmatrix}2\\5\end{pmatrix}$ $\vec{b} = \begin{pmatrix}-3\\4\end{pmatrix}$

a. $|a+b|$

$= \begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}-3\\4\end{pmatrix}$

$= \begin{pmatrix}-1\\9\end{pmatrix}$

$|a+b| = \sqrt{(-1)^2 + (9^2)}$

$= \sqrt{82}$

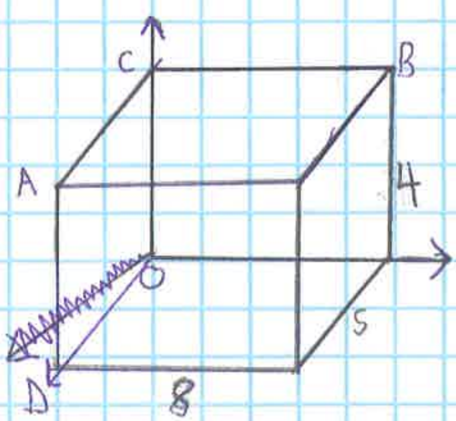
b. Write the component form of $\vec{a} - 4\vec{b}$

$\begin{pmatrix}2\\5\end{pmatrix} - 4\begin{pmatrix}-3\\4\end{pmatrix}$

$\begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-12\\16\end{pmatrix}$

$\begin{pmatrix}14\\-11\end{pmatrix}$

3.



c. perpendicular vector to ΔCDB?

$\vec{CD} \times \vec{DB}$

$\begin{vmatrix} i & j & k \\ 5 & 0 & -4 \\ -5 & 8 & 4 \end{vmatrix}$

$\begin{pmatrix} 0-(-32) \\ -(20-20) \\ 40-(0) \end{pmatrix}$

$\begin{pmatrix}32\\0\\40\end{pmatrix}$

$32i + 40k$

a. $\vec{CD}$

C: $\begin{pmatrix}0\\0\\4\end{pmatrix}$ $\begin{pmatrix}5\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\4\end{pmatrix}$

D: $\begin{pmatrix}5\\0\\0\end{pmatrix}$ $\vec{CD} = \begin{pmatrix}5\\0\\-4\end{pmatrix}$

b. ~~$\vec{BD}$~~

~~B: $\begin{pmatrix}0\\8\\4\end{pmatrix}$ $\begin{pmatrix}5\\0\\0\end{pmatrix} - \begin{pmatrix}0\\8\\4\end{pmatrix}$~~

~~D: $\begin{pmatrix}5\\0\\0\end{pmatrix}$ $\begin{pmatrix}5\\-8\\-4\end{pmatrix}$~~

$\vec{DB}$ $\begin{pmatrix}0\\8\\4\end{pmatrix} - \begin{pmatrix}5\\0\\0\end{pmatrix}$

B: $\begin{pmatrix}0\\8\\4\end{pmatrix}$ $\begin{pmatrix}-5\\8\\4\end{pmatrix}$

D: $\begin{pmatrix}5 \end{pmatrix}$

d. area?

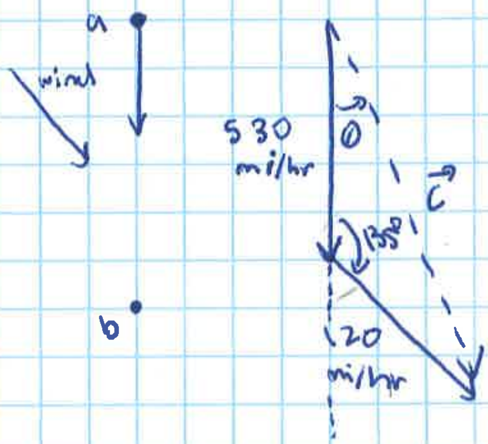
area $= \frac{1}{2} |\vec{CD} \times \vec{DB}|$

$= \frac{1}{2} \sqrt{32^2 + 40^2}$

$= 25.6$ units^2

4. An Airplane is attempting to fly from point a to point b at a speed of 530 mi/hr. There is a wind blowing towards the southeast of 120 mi/hr.

a. What is the airplane's actual speed and direction?



$\vec{c}^{\,2} = a^2 + b^2 - 2(a)(b)\cos(\theta)$

$c = \sqrt{530^2 + 120^2 - 2(530)(120)\cos(135)}$

$c = \sqrt{295,300 - 127200\cos(135)}$

$\boxed{\text{actual speed} = 621 \text{ mi/hr}}$

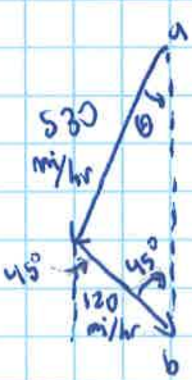
$$\frac{\sin(135)}{621} = \frac{\sin(\theta)}{120}$$

$120\sin(135) = 621(\sin(\theta))$

$$\theta = \sin^{-1}\left(\frac{120\sin(135)}{621}\right)$$

$\boxed{\theta = 7.85° \text{ SE}}$

b. What direction must the plane fly to compensate for the wind



$$\frac{\sin(45)}{530} = \frac{\sin(\theta)}{120}$$

$120\sin(45) = 530\sin(\theta)$

$$\theta = \sin^{-1}\left(\frac{120\sin(45)}{530}\right)$$

$\boxed{\theta = 9.21° \text{ SW}}$