Day Four : Calculus and Others

Math Induction proof process for a given conjecture (statement):
Step 1: Show that the statement is true for an initial case, n=1.
Step 2: Assume that the statement is true for n=k where $k \in Z^+$.
Step 3: Prove that the statement is true for n=k+1.
Step 4: ∴ The statement is true for $n \in Z^+$

Problem 1) IB Question

The function f is defined by $f(x) = e^x \sin x$.

(a) Show that $f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right)$.

(b) Obtain a similar expression for $f^{(4)}(x)$.

(c) Suggest an expression for $f^{(2n)}(x)$, $n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction.

a) $f'(x) = e^x \sin x + e^x \cos x$

$f^{2}(x) = f''(x) = 2e^x \cos x = 2e^x \sin(x + \frac{\pi}{2})$

b) $f'''(x) = 2e^x \sin(x + \frac{\pi}{2}) + 2e^x \cos(x + \frac{\pi}{2})$

$f^{4}(x) = 4e^x \cos(x + \frac{\pi}{2}) = 4e^x \sin(x + \pi) \Leftarrow \cos(x + \frac{\pi}{2}) = \sin(x + \pi)$

$= 2^2 e^x \sin(x + \frac{\pi \cdot 2}{2}) = 2^2 e^x \sin(x + \frac{\pi \cdot 2}{2})$

c) Conjecture

$f^{2n}(x) = 2^n e^x \sin(x + \frac{n\pi}{2})$ is true for $n \in Z^+$

Induction proof:

1) when $n = 1 \Rightarrow f''(x) = 2e^x \sin(x + \frac{\pi}{2})$

2) Assume $f^{2k}(x) = 2^k e^x \sin(x + \frac{k\pi}{2})$ is true for $n = k$ where $k \in Z^+$

3) when $n = k+1$,

$f^{2(k+1)}(x) = f^{2k+2}(x) = \cancel{2^k e^x \sin(x + \frac{k\pi}{2})} + 2^k e^x \cos(x + \frac{k\pi}{2})$

$+ 2^k e^x \cos(x + \frac{k\pi}{2}) - \cancel{2^k e^x \sin(x + \frac{k\pi}{2})} \rightarrow$

$$= 2^{k+1} e^{x} \cos\left(x + \frac{k\pi}{2}\right) = 2^{k+1} \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$$

$$= 2^{k+1} e^{x} \sin\left(x + \frac{\pi(k+1)}{2}\right)$$

4) $\therefore f^{(2n)}_{(x)} = 2^{n} e^{x} \sin\left(x \ \frac{n\pi}{2}\right)$ is true for $n \in \mathbb{Z}^{+}$

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Problem 2)

Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2, n \in \mathbb{Z}$.

proposition : $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$

1) Consider the proposition when $n=2$.

$1 + \frac{1}{\sqrt{2}} > \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}} > \sqrt{2} - 1 \Rightarrow$ The proposition is true for $n=2$.

2) Assume for $n=k$: $\sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ is true.

$\Rightarrow 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdots \frac{1}{\sqrt{k}} > \sqrt{k}$

3) If $n=k+1$

$\Rightarrow \underbrace{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdots \frac{1}{\sqrt{k}}} + \frac{1}{\sqrt{k+1}}$

$\Rightarrow \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} > \frac{\sqrt{k} \cdot \sqrt{k} + 1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$

$\therefore$ 4) $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ is true for $n \geq 2$.

Practice) Work on separate of paper.

1. Using Mathematical induction, prove $\frac{d^n y}{dx^n} = (-1)^{n-1} \cdot \frac{2(n-3)!}{(1+x)^{n-2}}$ **for** $n > 2$ *if* $y = (1+x)^2 \ln(1+x)$

2. A sequence is definded by the reccurence relation $x_n = x_{n-1} + x_{n-2}$ **for** $n \geq 3$ *and* $x_1 = 1$ *and* $x_2 = 2$.
   Prove, using mathematical induction, that $(\frac{1+\sqrt{5}}{2})^n > x_n$ for all $n \in Z^+$.