Day Four : Calculus and Others

Math Induction proof process for a given conjecture (statement):
Step 1: Show that the statement is true for an initial case, n=1.
Step 2: Assume that the statement is true for n=k where $k \in Z^+$.
Step 3: Prove that the statement is true for n=k+1.
Step 4: ∴ The statement is true for $n \in Z^+$

Problem 1) IB Question

The function f is defined by $f(x) = e^x \sin x$.

(a) Show that $f''(x) = 2e^x \sin\left(x + \frac{\pi}{2}\right)$.

(b) Obtain a similar expression for $f^{(4)}(x)$.

(c) Suggest an expression for $f^{(2n)}(x)$, $n \in \mathbb{Z}^+$, and prove your conjecture using mathematical induction.

a) $f'(x) = e^x \sin x + e^x \cos x$

$f^{2}(x) = f''(x) = 2e^x \cos x = 2e^x \sin(x + \frac{\pi}{2})$

b) $f'''(x) = 2e^x \sin(x+\frac{\pi}{2}) + 2e^x \cos(x+\frac{\pi}{2})$

$f^{4}(x) = 4e^x \cos(x+\frac{\pi}{2}) = 4e^x \sin(x+\pi)$ ⇐ $\cos(x+\frac{\pi}{2}) = \sin(x+\pi)$

$= 2^2 e^x \sin(x + \frac{\pi \cdot 2}{2}) = 2^2 e^x \sin(x + \frac{\pi \cdot 2}{2})$

c) Conjecture

$f^{2n}(x) = 2^n e^x \sin(x + \frac{n\pi}{2})$ is true for $n \in Z^+$

Induction proof:

1) when $n=1$ ⇒ $f''(x) = 2e^x \sin(x+\frac{\pi}{2})$

2) Assume $f^{2k}(x) = 2^k e^x \sin(x + \frac{k\pi}{2})$ is true for $n=k$ where $k \in Z^+$

3) when $n = k+1$,

$f^{2(k+1)}(x) = f^{2k+2}(x) = 2^k e^x \cancel{\sin(x+\frac{k\pi}{2})} + 2^k e^x \cos(x+\frac{k\pi}{2})$
$+ 2^k e^x \cos(x+\frac{k\pi}{2}) - 2^k e^x \cancel{\sin(x+\frac{k\pi}{2})}$ →

$= 2^{k+1} e^x \cos(x + \frac{k\pi}{2}) = 2^{k+1} \sin(x + \frac{k\pi}{2} + \frac{\pi}{2})$

$= 2^{k+1} e^x \sin(x + \frac{\pi(k+1)}{2})$

4) $\therefore f^{(2n)}(x) = 2^n e^x \sin(x \ \frac{n\pi}{2})$ is true for $n \in \mathbb{Z}^+$

(14)

Problem 2)

Prove, by mathematical induction, that $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ for $n \geq 2, n \in \mathbb{Z}$.

proposition : $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$

1) Consider the proposition when $n=2$.

$1+\frac{1}{\sqrt{2}} > \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}} > \sqrt{2}-1 \Rightarrow$ The proposition is true for $n=2$.

2) Assume for $n=k$: $\sum_{r=1}^{r=k} \frac{1}{\sqrt{r}} > \sqrt{k}$ is true.

$\Rightarrow 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} \cdots \frac{1}{\sqrt{k}} > \sqrt{k}$

3) If $n=k+1$

$\Rightarrow \underbrace{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} \cdots \frac{1}{\sqrt{k}}} + \frac{1}{\sqrt{k+1}}$

$\Rightarrow \sqrt{k}+\frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} > \frac{\sqrt{k}\cdot\sqrt{k}+1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$

$\therefore$ 4) $\sum_{r=1}^{r=n} \frac{1}{\sqrt{r}} > \sqrt{n}$ is true for $n \geq 2$.

Practice) Work on separate of paper.

1. Using Mathematical induction, prove $\frac{d^n y}{dx^n} = (-1)^{n-1} \cdot \frac{2(n-3)!}{(1+x)^{n-2}}$ **for** $n > 2$ *if* $y = (1+x)^2 \ln(1+x)$

2. A sequence is definded by the reccurence relation $x_n = x_{n-1} + x_{n-2}$ **for** $n \geq 3$ *and* $x_1 = 1$ *and* $x_2 = 2$.

   Prove, using mathematical induction, that $(\frac{1+\sqrt{5}}{2})^n > x_n$ for all $n \in Z^+$.

Day four practice W.S

#1. $p(n)$: $\dfrac{d^n y}{dx^n} = (-1)^{n-1} \cdot \dfrac{2(n-2)!}{(1+x)^{n-2}}$ where $y = (1+x)^2(\ln(1+x))$

$n > 2$.

1) When $n=3$

L.H.S

$\dfrac{dy}{dx} = 2(1+x)\ln(1+x) + (1+x)$

$\dfrac{d^2y}{dx^2} = 2\ln(1+x) + 3$

$\dfrac{d^3y}{dx^3} = \dfrac{2}{1+x}$.

R.H.S.

$(-1)^{3-1} \cdot \dfrac{2(3-2)!}{(1+x)^{3-2}}$

$= \dfrac{2}{1+x}$.

L.H.S = R.H.S $\Rightarrow$ $p(n)$ is true for $n=3$

2)

When $n=k$ where $k \in \mathbb{Z}^+$ and $k > 2$ $\Rightarrow$ Assume. $\dfrac{d^k y}{dx^k} = (-1)^{k-1} \dfrac{2(k-2)!}{(1+x)^{k-2}}$ is true.

3) If $n = k+1$, $\dfrac{d^{k+1}y}{dx^{k+1}} = \dfrac{d}{dx}\left[\dfrac{(-1)^{k-1} \cdot 2(k-2)!}{(1+x)^{k-2}}\right] = (-1)^{k-1} \cdot 2(k-2)! \dfrac{d}{dx}\left[(1+x)^{2-k}\right]$

$= (-1)^{k-1} \cdot 2(k-2)!\,(2-k)(1+x)^{1-k}$

$= (-1)^{k-1} \cdot 2(k-2)!\,(-1)(k-2)(1+x)^{1-k}$

$= (-1)^{(k+1)-1} \cdot 2\left[((k+1)-2)!\right](1+x)^{-[(k+1)-2]}$

$= \boxed{\dfrac{(-1)^{(k+1)-1} \cdot 2[(k+1)-2]!}{(1+x)^{(k+1)-2}}}$

4) $\therefore$ $p(n)$ is true for $n > 2$.

3 proposition : $\left.\begin{cases} x_n = x_{n-1} + x_{n-2} & \text{for } n \geq 3 \\ x_1 = 1 \\ x_2 = 2. \end{cases}\right\} \Rightarrow \left(\frac{1+\sqrt{5}}{2}\right)^n > x_n.$

1) When $n=3$.

R.H.S. $x_3 = x_2 - x_1 = 2-1 = 1$

L.H.S $\left(\frac{1+\sqrt{5}}{2}\right)^1$

$\frac{1+\sqrt{5}}{2} > 1$ $\Rightarrow$ so the proposition is true for $n=3$.

2) When $n=k \Rightarrow$ Assume $\boxed{\left(\frac{1+\sqrt{5}}{2}\right)^k > x_k}$ is true.

3) When $n=k+1 \Rightarrow$ show $\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} > x_{k+1}$

$$\underbrace{x_{k+1} = x_k + x_{k-1}}_{\text{Recursive Definition.}} < \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2} + 1\right)$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left[\frac{1+\sqrt{5}+2}{2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right)$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}.$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$$

$$\Rightarrow \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} > x_{k+1}$$

4) $\therefore$ The proposition $\left(\frac{1+\sqrt{5}}{2}\right)^n > x_n$ for all $n \in \mathbb{Z}^+$