CHAPTER 17:

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Concepts:

- Informal Definition of a Limit
    - What y-value does f(x) get close to as x approaches c
        - close: as close as you need to be convinced of the result
        - approaches: closer/closer but not actually there
            - this means that for the purpose of this limit, we don't care what f(c) is or if it exists
- Notation: $\lim_{x \to c} f(x) = \_\_\_\_$
- $\lim_{x \to 0} \frac{\sin(n)}{n} = 1$
- Definition of Derivative: $\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$
    - Derivative of x: instantaneous rate of change of f(x)
- Continuity:
    - a function is continuous @ x = a if & only if
        1. f(a) exists
        2. $\lim_{x \to a} f(x)$ exists
        3. $\lim_{x \to a} f(x) = f(a)$

- $f'(x) = n x^{n-1}$

  ↑ Power Rule

- Addition Rules
  - If $f(x) = a \cdot g(x)$ then $f'(x) = a g'(x)$
  - If $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$
- Chain Rule
  - If $y = f(u)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
- Product Rule
  - If $f(x) = u(x) \cdot v(x)$ then $f'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$
- Quotient Rule
  - If $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{v(x) \cdot u'(x) - u(x) v'(x)}{(v(x))^2}$

(3.)

$f(x) = 2x + |x| \quad f(x) = \begin{cases} 4x, & x \geq 0 \\ 2x & x < 0 \end{cases}$

Prove is continuous at point (0,0)

$f(0) = 0$

$\lim_{x \to 0^+} \cancel{3}x = \lim_{x \to 0^-} x = 0$

$\therefore$ f(x) is continus at $x = 0$

④

$f(x) = \frac{6\sin x + 5\sin^2 x}{x}$

Evaluate $\lim_{x \to 0} f(x)$

$\lim_{x\to 0} f(x) = \lim_{x \to 0} \frac{6\sin x + 5\sin^2 x}{x}$

$= \lim_{x \to 0} \frac{\sin x}{x}(6 + 5\sin x)$

$= \lim_{x \to 0} 6 + 5\sin x$

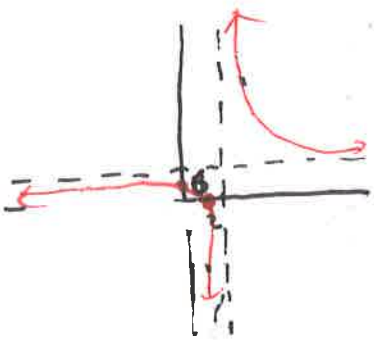
$= \boxed{6}$

3.) Find the limit of $\lim_{x \to 2^-} \frac{6x-1}{x-2}$

Asymp:

x: $x = 2$

y: $\frac{6x}{x} = 6$



$\therefore \lim_{x \to 2^-} \frac{6x-1}{x-2} = -\infty$

Definition of $f'(x)$: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

4. $f(x) = 6x^2 - 5x + 3$

Find Derivative of $f(x)$

Check: $f'(x) = (6)(2)(x^{2-1}) - (5)(1)(x^{1-1}) + (0)(3)(x^{0-1})$

$$\boxed{f'(x) = 12x - 5}$$

Solution Next page.

4) find the derivative of
$f(x) = 6x^2 - 5x + 3$

(5)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6(x+h)^2 - 5(x+h) + 3 - 6x^2 + 5x - 3}{h}$$

$$= \lim_{h \to 0} \frac{6(x^2 + 2hx + h^2) - \cancel{5x} - 5h + \cancel{3} - 6x^2 + \cancel{5x} - \cancel{3}}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{6x^2} + 12hx + 6h^2) - 5h - \cancel{6x^2}}{h}$$

$$= \lim_{h \to 0} \frac{12hx + 6h^2 - 5h}{h}$$

$$= \lim_{h \to 0} 12x + 6h - 5$$

$$= 12x - 5$$

Chapter 18: Rules of Differentiation

1

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{OR} \quad f'(x)^n = n \cdot f(x)^{n-1} \cdot f'(x)$$

Product Rule

If $f(x) = u(x) \cdot v(x)$ then,

$$\frac{df}{dx} = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$$

Quotient Rule

If $f(x) = \frac{u}{v}$ then,

$$\frac{df}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Implicit Differentiation

If $y = x + y$ then,

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dy}(y)$$

$$\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

Derivatives of Exponential Functions

If $f(x) = a^{nx}$ then, $f'(x) = \frac{d}{dx}(nx) \cdot \ln a \cdot a^x$

$$\frac{d}{dx}(e^x) = \overset{1}{\frac{d}{dx}(x)} \cdot \overset{1}{\ln e} \cdot e^x$$

$$e^x$$

Derivatives of Log Functions

If $f(x) = \log_a b(x)$ then, $f'(x) = \frac{1}{b(\ln a)} \cdot b'(x)$

If $f(x) = \ln(b(x))$ then, $f'(x) = \frac{1}{b} \cdot b'(x)$

OR $(\log_a b^x)' = \frac{1}{b \cdot \ln a} \cdot b^x$

Derivatives of Trig Functions

If $f(x) = \sin x$, $f'(x) = \cos x \cdot x'$

If $f(x) = \cos x$, $f'(x) = (-\sin x) \cdot x'$

If $f(x) = \tan x$, $f'(x) = \sec^2 x \cdot x'$

If $f(x) = \sec x$, $f'(x) = \sec x \cdot \tan x \cdot x'$

If $f(x) = \csc x$, $f'(x) = (-\csc x) \cdot \cot x \cdot x'$

If $f(x) = \cot x$, $f'(x) = (-\csc^2 x) \cdot x'$

1. $x^2 \cdot e^{2x^2+1} + 4y^2 = 1$

$$2x \cdot e^{2x^2+1} + x^2 \cdot 4x \cdot e^{2x^2+1} + 8y \cdot y' = 0$$

$$y' = \frac{-\cancel{2}x e^{2x^2+1} - \cancel{4}x^3 e^{2x^2+1}}{\cancel{8}y}$$

$$y' = \frac{-xe^{2x^2+1} - 2x^3 e^{2x^2+1}}{4y}$$

2. $h(x) = \frac{e^x}{x^2}$

$$h'(x) = \frac{(e^x)'(x^2) - (e^x)(x^2)'}{(x^2)^2}$$

$$h'(x) = \frac{e^x(x^2) - (e^x)(2x)}{x^4}$$

$$h'(x) = \frac{e^x[x^2 - 2x]}{x^4}$$

$$h'(x) = \frac{e^x(x-2)}{x^3}$$

3. $g(x) = \log_5 3x^2$

If $f(x) = \log_a b(x)$

$$f'(x) = \frac{1}{b(x)(\ln a)} \cdot b'(x)$$

$$g'(x) = \frac{1}{3x^2(\ln 5)} \cdot 6x$$

$$g'(x) = \frac{6x}{3x^2(\ln 5)}$$

$$g'(x) = \frac{2}{x(\ln 5)}$$

DERIVATIVES OF INVERSE TRIG

③

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

REMEMBER:

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

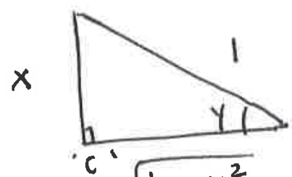
$\cot\theta = \frac{1}{\tan\theta}$

→ show $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ where $x \in [-1, 1]$

$y = \sin^{-1}x \rightarrow x = \sin y$

$\frac{d}{dx}x = \frac{d}{dx}(\sin y)$

$1 = \cos y\,(y')$



$1 = x^2 + c^2$

$c = \sqrt{1-x^2}$

$\sqrt{1-x^2} = \cos y$

$y' = \frac{1}{\cos y} \rightarrow \frac{1}{\sqrt{1-x^2}}$

→ note: since derivatives of trig end up being the other form derivative(?) use the info booklet.

1) solve:

$y = \cos(3x+1)$

$\frac{dy}{dx} = \sin(3x+1)(3)$

2) $y = \sin^{-1}(\sqrt[3]{2x})$ → $\csc(\sqrt[3]{2x})$

$\frac{dy}{dx} = -\csc(\sqrt[3]{2x}) \cdot \cot(\sqrt[3]{2x}) \cdot \frac{1}{3} \cdot (2x^{-2/3})(2)$

$\frac{dy}{dx} = -\csc(2x^{\frac{1}{3}}) \cdot \cot(2x^{\frac{1}{3}}) \cdot \frac{1}{3} \cdot 2 \cdot (2x^{-\frac{2}{3}})$

Do Not use this

3) $y = (\cot\sqrt{x})^4$

$\frac{dy}{dx} = 4(\cot\sqrt{x})^3(-\csc^2(\sqrt{x}))(\frac{1}{2})(x^{-\frac{1}{2}})$

$\frac{dy}{dx} = \frac{2(\cot\sqrt{x})^3(-\csc^2\sqrt{x})}{\sqrt{x}}$

2) $y = \sin^{-1}(\sqrt{2x})$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{2x})^2}} \cdot \frac{1}{2}(2x)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{1-2x} \cdot \sqrt{2x}}$

IMPLICIT DIFFERENTIATION

When an equation is not a function, relationships between x and y is <u>implicit</u>

To find $\frac{dy}{dx}$ you have to use "implicit differentiation"

e.g. $\frac{d}{dx}(y^3)$

$\frac{d}{dy}(y^3) \cdot \frac{dy}{dx}$ — this is the full notation and can be omitted while solving

$3y^2 \cdot \frac{dy}{dx}$

$$\therefore \frac{d}{dx}(y^n) = n(y^{n-1})\left(\frac{dy}{dx}\right)$$

if the variable isn't x, it needs the rate of change (e.g. $\frac{du}{dx}$)

PRACTICE

Find $\frac{dy}{dx}$:

1) $x^2 + y^2 = 4254353686$

2) $3x^2 - 3xy - 713y^2 = 6x$

SOLUTIONS

1) $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(4254353686)$

$2x + 2y\left(\frac{dy}{dx}\right) = 0$

$2y\left(\frac{dy}{dx}\right) = -2x$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

2) $\frac{d}{dx}(3x^2 - 3xy - 713y^2) = \frac{d}{dx}(6x)$

$\frac{d}{dx}3x^2 - 3\left[\left(\frac{d}{dx}x\right)(y) + (x)\left(\frac{d}{dx}\right)(y)\right] - \frac{d}{dx}713y^2 =$

$6x - 3\left[\left(1 \cdot y + x \cdot \frac{dy}{dx}\right)\right] - 1426y\frac{dy}{dx} = 6$

$6x - 3\left[y + x\frac{dy}{dx}\right] - 1426y\frac{dy}{dx} = 6$

$6x - 3y - 3x\frac{dy}{dx} - 1426y\frac{dy}{dx} = 6$

$-3x\frac{dy}{dx} - 1426y\frac{dy}{dx} = 6 - 6x - 3y$

$\frac{dy}{dx}(-3x - 1426y) = 6 - 6x - 3y$

$$\boxed{\frac{dy}{dx} = \frac{6-6x-3y}{-3x-1426y}}$$

LOGARITHMIC DIFFERENTIATION

 woah

Nile Camai 1P4
with consulting
from Blake
Abrahamsen

steps: ① take the natural log of everything
② differentiate

tips: remember $(\ln x)' = \frac{1}{x} \cdot x'$
the chain rule
log rules

$$y = \frac{e^{4x}(3x-2)^4(x^4+3)}{2-x}$$

$$\ln y = 4x(\cancel{\ln e}) + 4\ln(3x-2) + \ln(x^4+3) - \ln(2-x)$$

now we differentiate

chain rule!

$$\frac{dy}{dx} \cdot \frac{1}{y} = 4 + 4\left(\frac{1}{3x-2}\right)(3x-2)' + \left(\frac{1}{x^4+3}\right)(x^4+3)' - \left(\frac{1}{2-x}\right)(2-x)'$$

$$\frac{dy}{dx} \cdot \frac{1}{y} = 4 + \frac{12}{3x-2} + \frac{4x^3}{x^4+3} + \frac{1}{2-x}$$

can take shortcuts instead of showing every step

$$\therefore \frac{dy}{dx} = y\left[4 + \frac{12}{3x-2} + \frac{4x^3}{x^4+3} + \frac{1}{2-x}\right]$$

can plug in what we know:

$$\therefore \frac{dy}{dx} = \left[\frac{e^{4x}(3x-2)^4(x^4+3)}{2-x}\right]\left[4 + \frac{12}{3x-2} + \frac{4x^3}{x^4+3} + \frac{1}{2-x}\right]$$

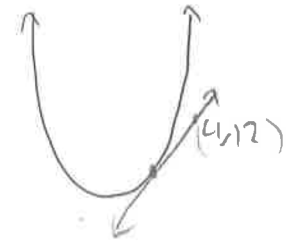
Equations of Tangents and Normals

3) Find the equations of the tangent to $f(x)=x^2$ from external point $(4,12)$.

(6)

→ Do not want to use x and y.

1) (a,a^2) on $f(x)=x^2 \to f'(x)=2x$
$f'(a)=2a \to$ slope



Find tangent line equation at (a,a^2)

$y-a^2=2a(x-a)$

2) $y-a^2=2a(x-a)$ must pas $(4,12)$

$12-a^2=2a(4-a) \to 12-a^2=8a-2a^2$

$a^2-8a+12=0$

$(a-6)(a-2)=0$

3) When $a=6$ $\boxed{y-36=2(x-6)}$

$a=2$ $\boxed{y-4=2(x-2)}$

Normal lines and tangent lines:

Find the equation of tangent line and normal line to graph of $f(x)=\sqrt{6x}$ at $x=6$

1) Find $\frac{df}{dx} \to \sqrt{6}\cdot\frac{1}{2}x^{-1/2} \to \frac{df}{dx}\Big|_{x=6}=\frac{\cancel{\sqrt{6}}}{2\cancel{\sqrt{6}}}=\frac{1}{2}$

2) Find $y \to f(6)=\sqrt{6\cdot 6}=6$
when $x=6$

$\therefore$ tangent line: $(6,6)$ $m=1/2$ $\boxed{y-6=\frac{1}{2}(x-6)}$

Normal line: $(6,6)$ $m=2$ $\boxed{y-6=-2(x-6)}$

SECOND ORDER IMPLICIT DIFFERENTIATION

- implicitly differentiate once, find $\frac{dy}{dx}$, then differentiate again
  - plug in the value for $\frac{dy}{dx}$ wherever it appears in solving for the second order

$$y^2 - x^2 = 4$$

$$\frac{d}{dx} y^2 - \frac{d}{dx} x^2 = \frac{d}{dx} 4$$

$$2y \frac{dy}{dx} - 2x = 0$$

$$2y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(x \cdot y^{-1}\right)$$

$$= 1 \cdot y^{-1} + x \cdot (-1) \cdot (y)^{-2} \cdot \frac{dy}{dx}$$

$$= y^{-1} - x \cdot y^{-2} \cdot \frac{x}{y}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{y} - \frac{x^2}{y^3}}$$

MVT: $f'(c) = \frac{f(b) - f(a)}{b - a}$ where $[a, b]$ is an interval and c is a number in that interval
(mean value theorem)

Rolle's Theorem: If a function is continuous on the interval $[a, b]$ and also differentiable, and $f(a) = f(b)$, then $f'(c) = 0$ where c is a value in $[a, b]$.

Continuity: 1) $f(a)$ is defined 2) $\lim_{x\to a} f(x)$ must exist 3) $f(a) = \lim_{x\to a} f(x)$

↳ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Differentiability: 1) f is continuous at $x = a$ 2) $\lim_{x\to a^+} f'(x) = \lim_{x\to a^-} f'(x)$

Differentiable = continuous

Continuous ≠ differentiable

Kinematics:

Distance: $s(t)$

Velocity: $v(t) = \frac{ds}{dt}$

↳ Avg. Velocity (speed): $|v(t)|$

Acceleration: $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Maximum Height or Distance: $v(t) = 0$, solve for t

↓

$s(t)$ = answer

Total Distance:

Absolute Value wherever it changes direction, add them all up.

[redacted]

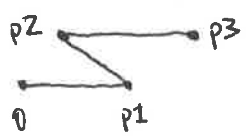
p1 = point where $s(t)$ changes direction, where $v(t)$ changes sign ($\pm$)

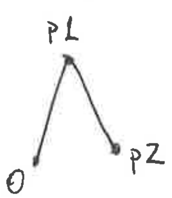
$|s(p1) - s(0)| + |s(p2) - s(p1)| + |s(p3) - s(p2)| + \ldots$

Do that for however many changes in direction there are, up until it stops moving.

Ex.

[redacted]





Ethan Zhu, Aniket Tripathy, Sam Zu, Elizabeth Bruski

Determine whether MVT can be applied to $f(x)=\frac{1}{x^2}$ on the interval [1, 6] 9

If it can be, find a value of c on the given interval

$$\frac{f(b)-f(a)}{b-a}$$

$f(x)=\frac{1}{x^2}$

$f'(x)=(x^{-2})'=-2x^{-3}=\frac{-2}{x^3}$

$$\frac{f(6)-f(1)}{6-1}=\frac{\frac{-2}{6^3}-\frac{2}{1^3}}{6-1}=\frac{\frac{-2}{216}+\frac{2}{1}}{6-1}$$

$$=\frac{\frac{-2}{216}+2}{5}=\frac{\frac{-430}{216}}{5}=\frac{86}{216}=\frac{43}{108}=f'(c)$$

$$\frac{-2}{x^3}=\frac{43}{108}$$

$$\frac{-2x^{-3}}{-2}=\frac{\frac{43}{108}}{-2}$$

$$x^{-3}=\frac{-43}{216}$$

$$\sqrt[3]{x^{-3}}=\sqrt[3]{\frac{-43}{216}}$$

$$x=-0.584$$

2

Given

$f(x)=2x+|x|$

prove that f is continuous but not differentiable at the point (0,0)

①

$$\begin{cases} 2x+x \\ x\geq 0 \\ y=3x \\ 2x-x \\ x<0 \\ x=y \end{cases}$$

$\lim_{x\to 0^+}=0$

$\lim_{x\to 0^-}=0$

$\therefore$ continuous

② differentiability

$(3x)'=3$

$(x)'=1$

$\lim_{x\to 0^+}(f'(3x))=-3$

$\lim_{x\to 0^-}(f'x)=1$

$\therefore$ not differentiable

1. (kinematics) 10

A ball thrown vertically upward at time $t = 0$ sec has height [redacted]

$h(t) = -12t^2 + 66t + 36$ in feet   3 sigfig answers

a. Find the ball's velocity and acceleration at time t

$v(t) = -24t + 66$

$a(t) = -24$

b. What [redacted] is the maximum height attained by the ball and at what time?

$v(t) = 0$

$-24t + 66 = 0$

$\frac{66}{24} = \frac{24t}{24}$

$\frac{11}{4} = t$

$\boxed{2.75s = t}$

$h(2.75) = -12(2.75)^2 + 66(2.75) + 36$

$= 126.75$

$\boxed{\approx 127 ft}$

c. Find the time when the ball hits the ground

$h(t) = 0$

$\frac{-12t^2 + 66t + 36 = 0}{6} \quad \frac{}{\emptyset}$

$-2t^2 + 11t + 6 = 0$

$-(2t^2 - 11t - 6) = 0$

$-(2t+1)(t-6) = 0$

$\cancel{t = -1/2}$ $\boxed{t = 6s}$

d. What was the velocity of the ball when it hit the ground?

$v(6) = -24(6) + 66$

$= -144 + 66$

$\boxed{= -78 ft/s}$

e. What is the total distance the ball travels?

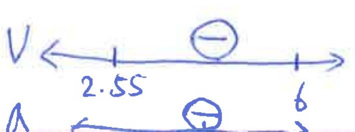
$|h(2.75) - h(0)| + |h(6) - h(2.75)| = 91 + 127 = \boxed{218 ft}$

$127 - 36 \qquad 0 - 127$

$-12(6)^2 + 66(6) + 36 = 0$

f. Determine if the speed is increasing or decreasing on time $(2.75, 6)$

⇒ Sign diagram



⇒ $V < 0$ and $a < 0$

⇒ speed increasing.