Chapter 19 Curve Analysis

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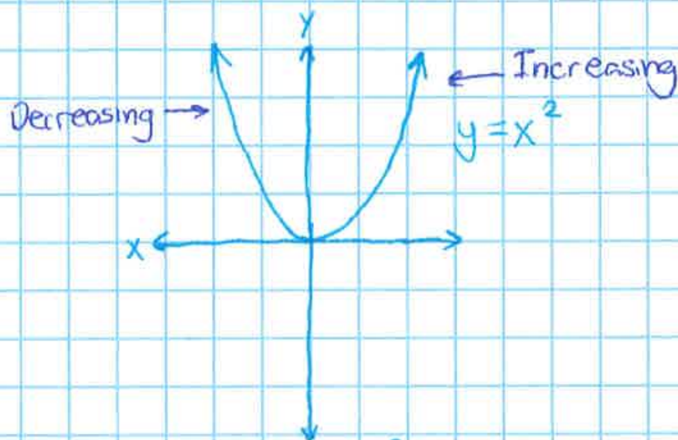
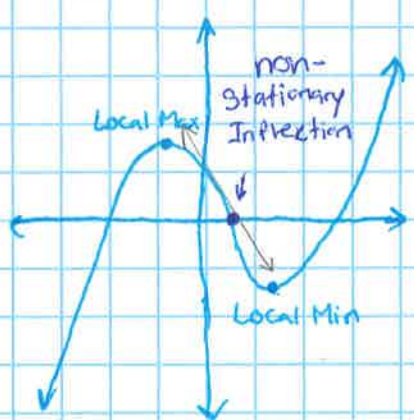
Major Concepts:

Increasing + Decreasing Functions:



when $x < 0$, the function is decreasing.
when $x > 0$, the function is increasing.

Inflection Points:



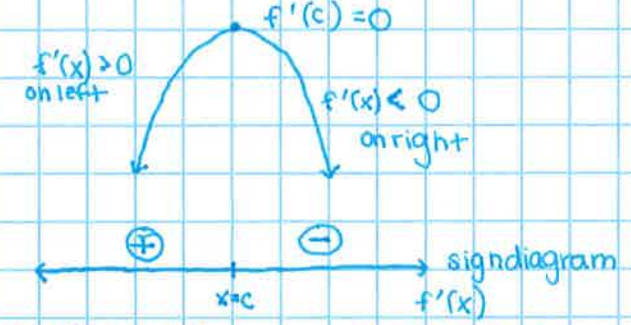
If a tangent at a point of inflection is horizontal, then the point is a stationary inflection.

If a tangent is not horizontal, then the point is a non-stationary inflection.

The First Derivative Test

The Critical Point is a relative local max where $f'(c)=0$.

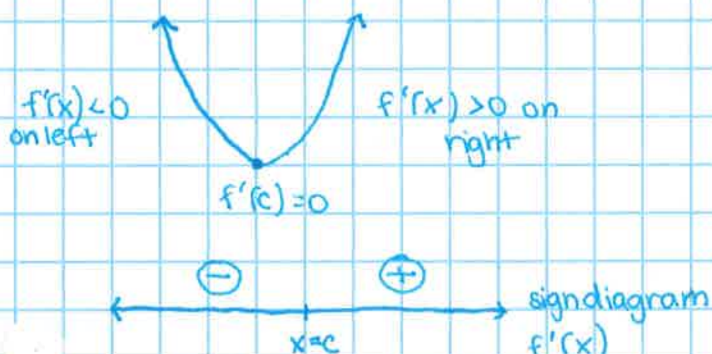
* where c is a root of the derivative function



the function is **increasing** on the left of the local max.

the function is **decreasing** on the right of the local max.

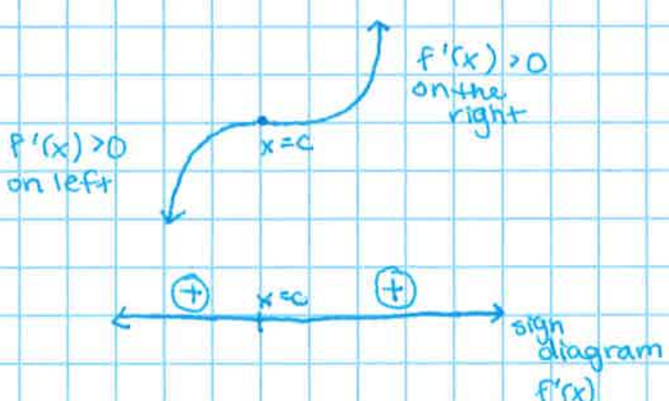
The Critical Point is a relative local min where $f'(c)=0$.



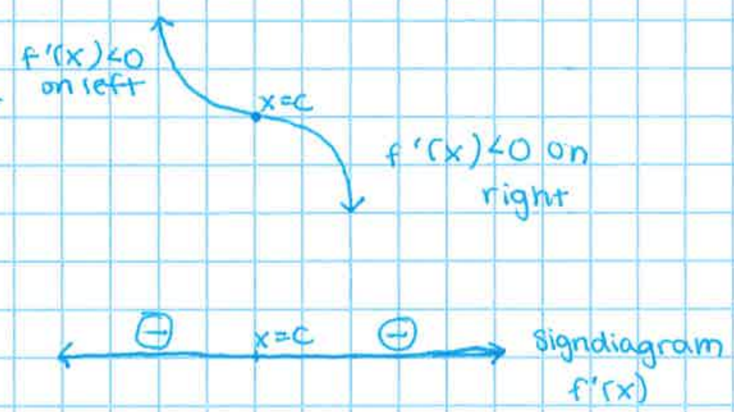
the function is **decreasing** on the left of the local min.

the function is **increasing** on the right of the local min.

The Critical Point is neither a relative local max or a relative min, where $f'(c)=0$.



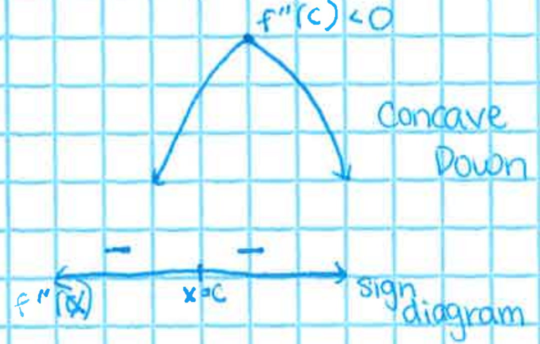
or



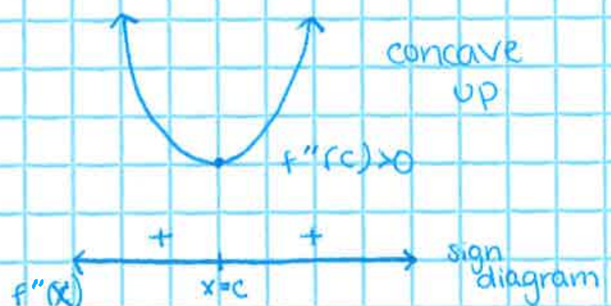
the function continues to inc/dec as the values continue to inc/dec

The Second Derivative Test

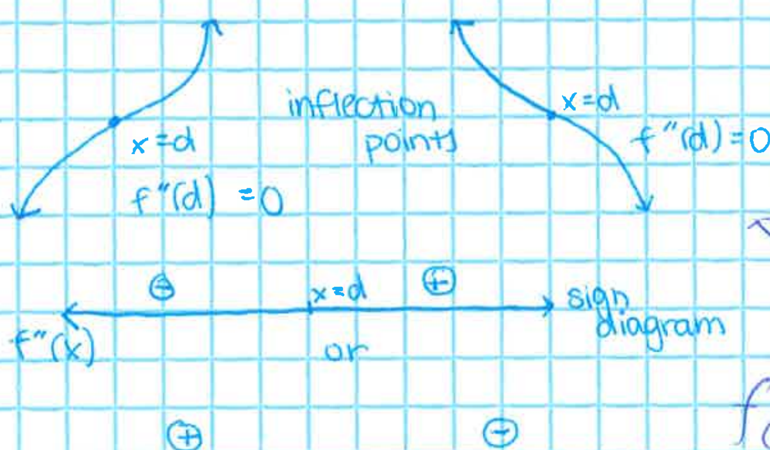
The Critical Point is a relative max if $f''(c) < 0$

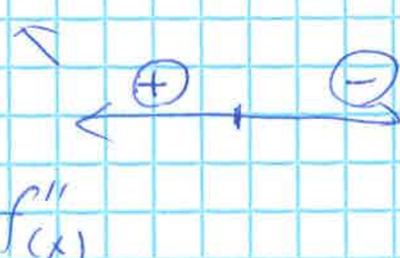


The Critical Point is a relative min if $f''(c) > 0$



if $f''(d) = 0$, then it is an inflection point

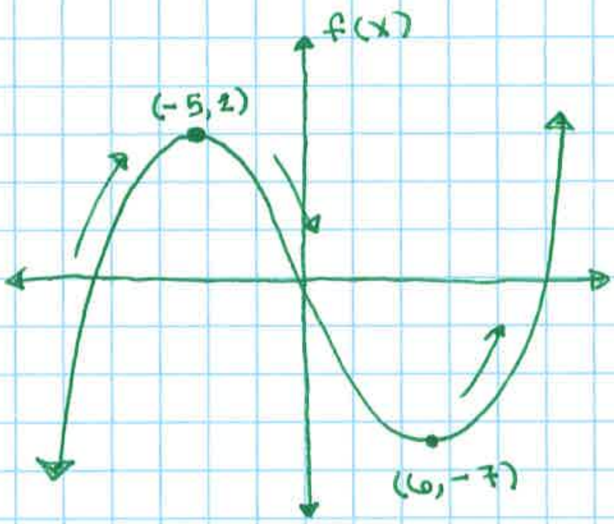




Example Problems

Example # 1:

On what intervals is $f(x)$ a. increasing
b. decreasing



a. $(-\infty, -5) \cup (6, \infty)$

b. $(-5, 6)$

Example #2:

$$f(x) = x^3 + 6x^2 + 3x$$

a) find the increasing and decreasing intervals

$$f'(x) = 3x^2 + 12x + 3$$

$$f'(x) = 3(x^2 + 4x + 1)$$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2}$$

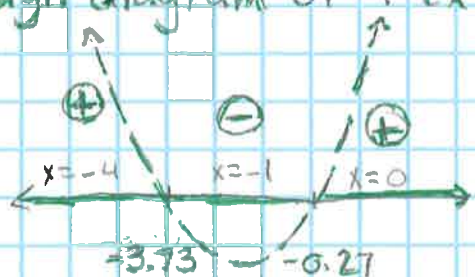
$$x = \frac{-4 \pm \sqrt{12}}{2}$$

$$x = \frac{-\cancel{4}^{2} \pm \cancel{2}\sqrt{3}}{\cancel{2}}$$

$$x = -2 \pm \sqrt{3}$$

$x \approx -0.27$ $\quad x \approx -3.73$

Sign diagram of $f'(x)$



$$f'(4) = 3(-4)^2 + 12(-4) + 3$$
$$= 48 - 48 + 3$$
$$= +3$$

Increasing: $(-\infty, -3.73) \cup (-0.27, \infty)$

$$f'(0) = 3(0)^2 + 12(0) + 3$$
$$= +3$$

Decreasing: $(-3.73, -0.27)$

$$f'(-1) = 3(-1)^2 + 12(-1) + 3$$
$$3 - 12 + 3 = -6$$

Example 3: $f(x)=3x^4-16x^3+24x^2-9$

a) Give the intervals where $f(x)$ is

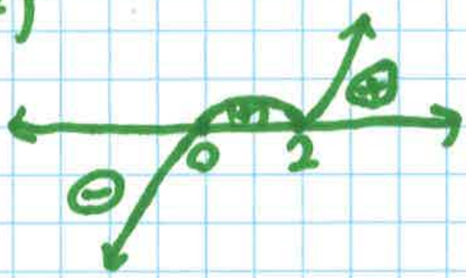
Increasing	Decreasing	Concave Up	Down
$(0,2)\cup(2,\infty)$	$(-\infty,0)$	$(-\infty,\frac{2}{3})\cup(2,\infty)$	$(\frac{2}{3},2)$

$f'(x)=12x^3-48x^2+48x$

$=12x(x^2-4x+4)$

$=12x(x-2)^2$

Stationary $\begin{cases} x=0 \\ x=2 \end{cases}$

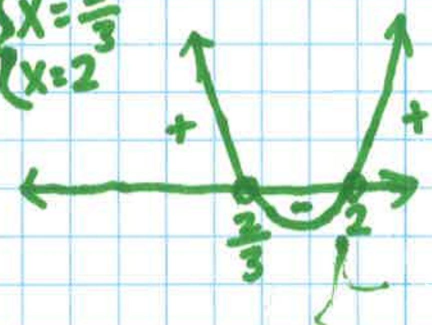


$f''(x)=36x^2-96x+48$

$=12(3x^2-8x+4)$

$=12(3x-2)(x-2)$

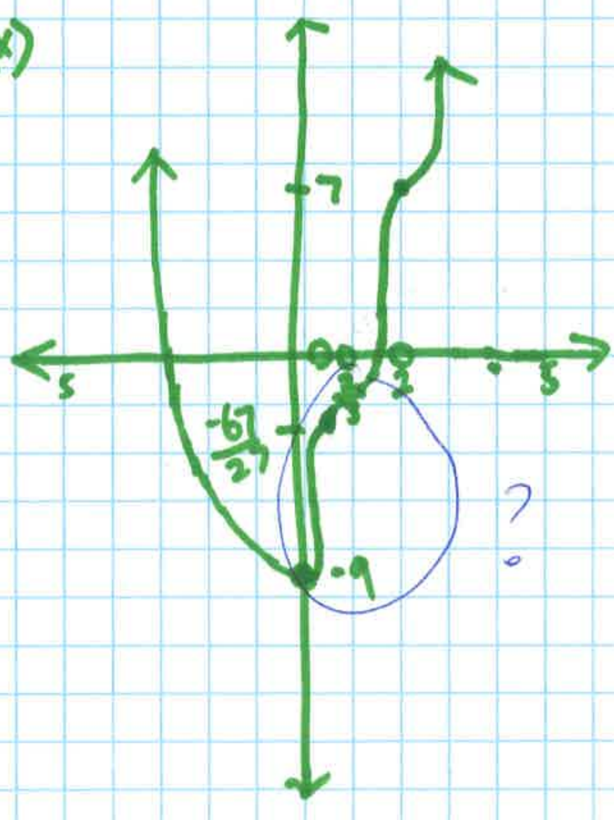
Inflection points $\begin{cases} x=\frac{2}{3} \\ x=2 \end{cases}$



b) Give coordinates where $f(x)$ has a

Local Max	Local Min	Stationary Inflection	Nonstationary Inflection
None	$(0,-9)$	$(2,7)$	$(\frac{2}{3},-\frac{67}{27})$

c) Sketch a graph of $f(x)$



Optimization and Related Rate

Gr. 6 (P. 4)

1. Draw a diagram of given situation (with appropriate notation).

2. Construct a formula with the variable to be optimized as the subject.

3. Find the first derivative and solve for x which makes the first derivative 0.

4. Confirm if the solution is the maximum or minimum and revisit if the solution is reasonable.

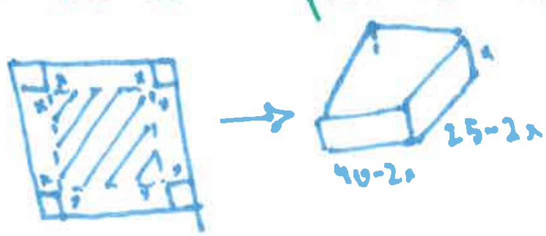
Related Rate

1. Draw the diagram and state the given information using proper rate change notation
2. Set up proper mathematical equation using variables
3. Differentiate the equation with respect to time (t)
4. Calculate the rate change using the given information

Optimization Problems

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate and then folding the metal to form the container. What size square must be cut to produce the cake dish of maximum volume?

1)



2) $V = w \cdot l \cdot h$ $\quad x \in (0, 12.5)$

$V = (40-2x)(25-2x)(x)$

3) $\frac{dv}{dx} = 0$ solve for x

$V = 4x^3 - 130x^2 + 1000x$

$\frac{dv}{dx} = 12x^2 - 260x + 1000 = 0$

$= 4(3x^2 - 65x + 250) = 0$

$= 4(3x - 50)(x - 5) = 0$

$\cancel{x = \frac{50}{3}}$ $\quad x = 5$

4) check:

$\frac{d^2v}{dx^2} = 24x - 260$

$\left.\frac{d^2v}{dx^2}\right|_{x=5} = (24)(5) - 260$

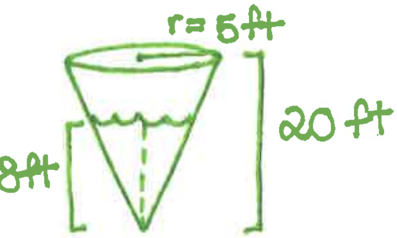
$= -140 < 0$

max. value

$\therefore$ 5cm by 5cm

Related Rate Problems

A tank with water is in the shape of an inverted cone 20 ft high with a circular base on the top whose radius is 5 ft. Water is running out of the bottom of tank at the constant rate of 18 ft^3/s. How fast is the water level falling when the water is 8 ft deep?



$\frac{dv}{dt} = 18\ ft^3/s \qquad \frac{dh}{dt} = ? \quad$ when $h = 8$

$$V = \frac{\pi r^2 h}{3} = \frac{\pi\left(\frac{h}{4}\right)^2 h}{3} = \frac{\frac{\pi h^2}{16}\cdot h}{3} = \frac{\pi h^3}{48}$$

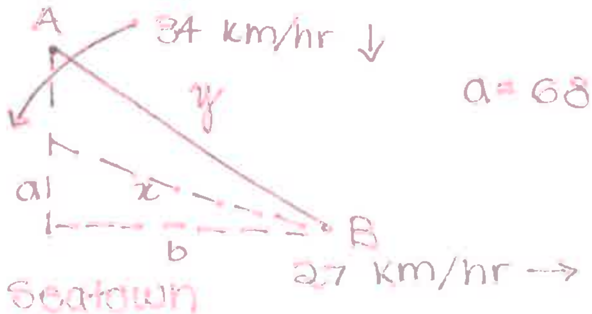
$\frac{h}{20} = \frac{r}{5} \qquad r = \frac{h}{4}$

$$\frac{dv}{dt} = \frac{\pi}{48}\cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi}{16} h^2 \cdot \frac{dh}{dt}$$

$$18 = \frac{\pi}{16}(8)^2 \cdot \frac{dh}{dt} \qquad 18 = 4\pi \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{18}{4\pi} = \frac{9}{2\pi}\ ft/s \qquad \boxed{\frac{dh}{dt} = \frac{9}{2\pi}\ ft/s}$$

Car A heads due South toward Seatown at 34 km/hr. Car B heads due east away from Seatown at 27 km/hr. Three hours after car B leaves Seatown, car A is still 68 km away from Seatown. How fast is the distance between the cars changing at this time?



$a = 68$

$d_B = 3 \cdot 27 = 81$ km

$d_A = 34 \cdot 3 = 102$ km

102 km + 68 km = 170 km

$\frac{da}{dt} = 34$ km/hr

$\frac{db}{dt} = 27$ km/hr

$$a^2 + b^2 = c^2 \longrightarrow 2a\cdot\frac{da}{dt} + 2b\cdot\frac{db}{dt} = 2c\cdot\frac{dc}{dt}$$

$$a\cdot\frac{da}{dt} + b\cdot\frac{db}{dt} = c\cdot\frac{dc}{dt} \qquad c = \sqrt{68^2 + 81^2} = 105{,}76$$

$$68\cdot(-34) + 27\cdot 81 = 105.76\cdot\frac{dc}{dt}$$

$$-2312 + 2187 = 105.76\cdot\frac{dc}{dt}$$

$$\boxed{\frac{dc}{dt} = -1.18\ km/hr}$$

Group 7: optimization & related rates

optimization:

strategy:
1. draw a diagram of given situation with appropriate notation
2. construct a formula with the variable to be optimized as the subject
3. find the 1^{st} derivative and solve for x which makes the first derivative zero.
4. confirm that the solution is a maximum or a minimum and revisit if solution isn't reasonable

related rate:

Strategy:
1. set up diagram and given info with related rate notation (define derivatives, identify what needs to be found)
2. set up mathematical equations of notations from step 1
3. differentiate with respect to the TIME variable $\left(\frac{d}{dt}\right)$
4. calculate the rate of change by plugging in

1 related rate practice problem #1

A ladder to get into a treehouse rests against a perfectly vertical tree. The ladder is 15 feet long. ~~[illegible]~~ It slides away from the base of the tree at a rate of 3ft/second.

a.) How fast is the angle between the top of the ladder and the tree changing when the angle is $\frac{\pi}{3}$ radians?

b.) How fast is the top of the ladder sliding down the ~~[illegible]~~ tree when the ladder is 5 feet away? (at base)

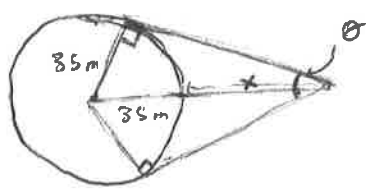
solution on back.

Each day, Jason arrives at work in his personal Formula 1 race car. Jason is an operator at a grain silo which is perfectly cylindrical shaped and has a diameter of 70 m. As Jason drives up to the edge of the grain silo in a straight line, he notices that θ, the angle of sight taken up the the silo in the horizontal direction, increases. When Jason is still 90 meters away from the closest point on the grain silo, how fast is θ changing, given that Jason's car has a speed of 50 m/s? Give your answer in rad/s.

Diagram of θ

Solution:

1) Make a diagram.

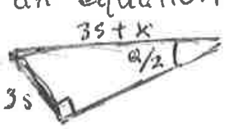


$r = 35\,m$ $\quad \dfrac{dx}{dt} = -50\ m/s$

$x = 90\,m$

$\dfrac{d\theta}{dt} = ?$

2) Write an equation:



$$\sin\left(\frac{\theta}{2}\right) = \frac{35}{35+x}$$

$$\frac{\theta}{2} = \sin^{-1}\left(\frac{35}{35+x}\right)$$

3) Differentiate w.r.t. TIME.

$$\theta = 2\sin^{-1}\left(\frac{35}{35+x}\right)$$

$$\frac{d\theta}{dt} = \left(2\cdot\frac{1}{\sqrt{1-\left(\frac{35}{35+x}\right)^2}}\right)\cdot\left(35\cdot\frac{-1}{(35+x)^2}\right)*\left(1\cdot\frac{dx}{dt}\right)$$

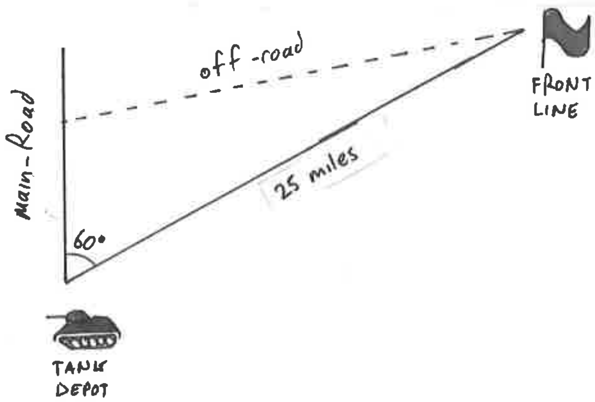
4) Evaluate: $x = 90\,m$, $\dfrac{dx}{dt} = -50\ m/s$

$$\frac{d\theta}{dt} = 2\cdot\frac{1}{\sqrt{1-\left(\frac{35}{125}\right)^2}}\cdot 35\cdot\frac{-1}{125^2}\cdot(-50)$$

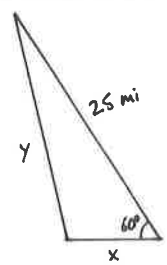
$$= \boxed{0.233\ \text{rad/s}}$$

It is 1943, and Pranav is a tank pilot in the Red Army. In order to reach the most ~~dangerous~~ active part of the front line, he must drive his tank along the road and at some point pass through a destroyed part of Stalingrad. On the road, he averages 0.662 gallons/mile, but offroad only gets 0.847 gallons/mile. Help comrade Pranav find the minimum amount of fuel (in gallons) possible to get his tank to the front line so he can ~~die~~ fight for Mother Russia.

Note: Fuel mileage data from tanks-encyclopedia.com, vehicle in question is a 1940-spec T34-76



① Make a drawing & list other known variables



$\theta = 60°$

② use law of cosines to express y in terms of x

$$y = \sqrt{x^2+25^2-2(x)(25)\cos 60°} = \sqrt{x^2+25^2-25x}$$

③ find gallons as an equation given gallons = mi $\cdot \frac{gal}{mi}$

$$\text{gallons} = d(g) = 0.662x + 0.847\sqrt{x^2-25x+25^2}$$

④ differentiate with respect to x

$$\frac{dg}{dx} = 0.662 + \frac{(0.847)(\frac{1}{2})(2x-25)}{\sqrt{x^2-25x+25^2}}$$

⑤ Set $\frac{dg}{dx}$ equal to zero and solve for x

$$0 = 0.662 + \frac{(0.847)(0.5)(2x-25)}{\sqrt{x^2-25x+25^2}}$$

$$\left(-0.662\sqrt{x^2-25x+25^2}\right)^2 = \left((0.847)(0.5)(2x-25)\right)^2$$

$$0.438(x^2-25x+25^2) = 0.179(4x^2-100x+25^2)$$

$$2.447x^2-61.175x+1529.375 = 4x^2-100x+25^2$$

$$-1.553x^2-38.825x+904.375$$

$$x = -39.68 \quad x = 14.68$$

negative = ignore

⑥ plug back into equation from ③ to find gallons

$$\text{gallons} = 0.662(14.68)+0.847\sqrt{14.68^2-(25\times14.68)+625} = \boxed{28.15 \text{ gallons}}$$

Varun, an officer in the Imperial Japanese Army, has received orders to construct a prison. This prison must contain 1.8 square kilometers. However, the Japanese are running low on resources and this prison must cost as little as possible. What dimensions would require the least fencing for this prison? [redacted]

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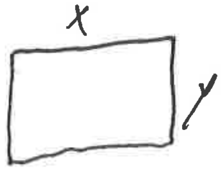


$A = x \cdot y = 1.8 \quad y = \frac{1.8}{x}$

$P = 2y + 2x$

$= \frac{3.6}{x} + 2x$

$\frac{dP}{dx} = 0$ solve for x

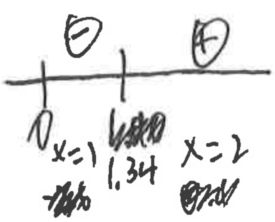
$\frac{dP}{dx} = \frac{-3.6}{x^2} + 2$

$\frac{-3.6}{x^2} = -2$

$2x^2 = -3.6$

$x^2 = 1.8$

$x \approx \pm 1.34$



1.34 is local minimum

$A = x \cdot y = 1.8$ ✓

$1.34y = 1.8$

$y = \frac{1.8}{1.34}$

$y = 1.34$

$\boxed{1.34 \text{ km} \times 1.34 \text{ Km}}$

CHAPTER 14: Vectors

scalar: ONLY magnitude

vector: BOTH magnitude and direction

length: $|a|$

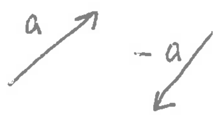
GEOMETRIC OPERATIONS

equality:

$a = b$ IF: ① a and b are in the same direction

② a and b have the same magnitude

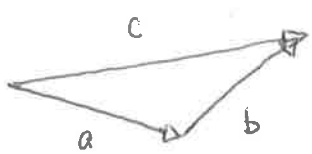
negativity:



$-a$ is negative of a IF: ① a and b are in ~~the~~ opposite directions

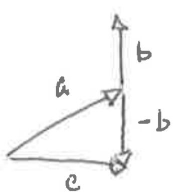
② a and b have the same magnitude

addition:



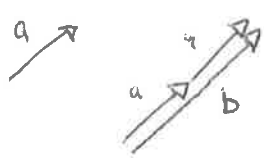
$a + b = c$

subtraction:



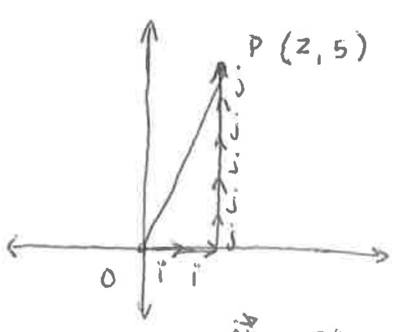
$a - b = c$

scalar multiplication:



$2a = b$

VECTORS IN PLANES AND SPACE

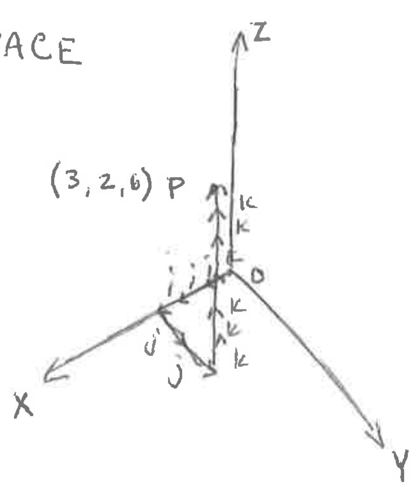


$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{OP} = 2i + 5j$

$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

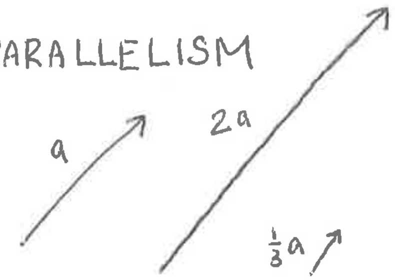


$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{OP} = 3i + 2j + 6k$

$\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

PARALLELISM



$a \parallel b$ iff one is a scalar multiple of the other

SCALAR / DOT PRODUCT

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \qquad W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$$

$$V \bullet W = V_1 W_1 + V_2 W_2 + V_3 W_3$$

APPLICATION

$$\cos\theta = \frac{V \bullet W}{|V|\,|W|}$$

where θ is the angle between V and W

CROSS PRODUCT

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

OR

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 b_2 - a_2 b_1)$$

APPLICATION

$$[OAB] = \frac{1}{2}|a \times b| = \frac{1}{2}|a||b| \sin\theta$$

where θ is the angle between a and b

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Given $\vec{OA} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ and $\vec{OB} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$, with θ being the angle between the two vectors, find $\tan\theta$.

$$\sin\theta = \frac{|a \times b|}{|a||b|}$$

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{|a \times b|}{|a||b|}}{\frac{a \cdot b}{|a||b|}} = \frac{|a \times b|}{a \cdot b}$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 0 & -5 \end{vmatrix} = i((3)(-5) - (-1)(0)) - j((1)(-5) - (-1)(2)) + k((1)(0) - (3)(2))$$

$$= -15i + 3j - 6k$$

$$|a \times b| = \sqrt{(-15)^2 + 3^2 + (-6)^2}$$

$$= \sqrt{270}$$

$$= 3\sqrt{30}$$

$$a \cdot b = (1)(2) + (3)(0) + (-1)(-5) = 7$$

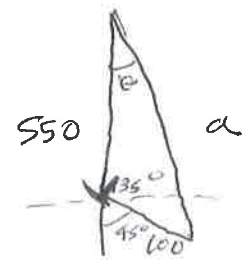
$$\tan\theta = \frac{3\sqrt{30}}{7}$$

$$\theta = \tan^{-1}\left(\frac{3\sqrt{30}}{7}\right)$$

$$\boxed{\theta = 66.9^\circ}$$

Suppose ~~Portland~~ Oakland is directly South of Vancouver. An airplane is attempting to fly due south to ~~Portland~~ Oakland. The plane's speed is 550 miles per hour. However, a wind is blowing at 100 miles/hr from the northwest. Find actual speed and direction from the south.

Solution:



Law of Cosines

$$a^2 = 550^2 + 100^2 - 2(550)(100)\cos 135^\circ$$

$$a = \sqrt{550^2 + 100^2 - 2(550)(100)\cos 135^\circ}$$

$$a \approx 624.725$$

$$\boxed{a = 625 \text{ m/hr}}$$

Law of sines

$$\frac{\sin 135^\circ}{625} = \frac{\sin\theta}{100}$$

$$\frac{100 \sin 135^\circ}{625} = \sin\theta$$

$$\sin^{-1}\left(\frac{100\sin 135^\circ}{625}\right) = \theta$$

$$\boxed{6.50^\circ = \theta}$$

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Consider the Points $A(-5,3,7)$, $B(2,4,2)$, and $C(0,-6,-1)$.

Find the Area of ΔABC.

$$\overrightarrow{BA} = \begin{pmatrix} -7 \\ -1 \\ 5 \end{pmatrix} \quad \overrightarrow{BC} = \begin{pmatrix} -2 \\ -10 \\ -3 \end{pmatrix}$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ -7 & -1 & 5 \\ -2 & -10 & -3 \end{vmatrix} = i(3+50) - j(21+10) + k(70-2) = 53i - 31j + 68k$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{pmatrix} 53 \\ -31 \\ 68 \end{pmatrix}$$

$$|\overrightarrow{BA} \times \overrightarrow{BC}| = \sqrt{(53)^2 - (31)^2 + (68)^2}$$

$$|\overrightarrow{BA} \times \overrightarrow{BC}| = \sqrt{2809 + 961 + 4624}$$

$$|\overrightarrow{BA} \times \overrightarrow{BC}| = \sqrt{8394}$$

$$|\overrightarrow{BA} \times \overrightarrow{BC}| = 91.6$$

$$A = \frac{1}{2} \cdot |\overrightarrow{BA} \times \overrightarrow{BC}|$$

$$A = \frac{1}{2} \cdot 91.6$$

$$\boxed{A = 45.8\ u^2}$$

Nico Lordi Cove 8