Part II: Alternating Series Remainder for the Sum:

If an alternating Series converges, the sum of the alternating series can be approximated with a remainder by

$$|S-S_n| = |R_n| \le a_{n+1}$$

Proof:

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k (-1)^{k+1}$$

$$S = a_1 - a_2 + a_3 - a_4 \cdots a_n + a_{n+1} \cdots a_\infty$$

$$-\underline{|\, S_n = a_1 - a_2 + a_3 - a_4 \cdots a_n}$$

$$S - S_n = a_{n+1} - a_{n+2} \cdots a_\infty$$

$$\therefore \boxed{|S-S_n| = |R_n| \le a_{n+1}}$$

Example 1) Approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$ for its first six terms.

$$\sum_{n=1}^{6}(-1)^{n+1}\frac{1}{n!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} \approx 0.63194.$$

$$\approx 0.632.$$

$$a_7 = \frac{1}{7!} = \frac{1}{5040}$$

$$0.63194 - \frac{1}{5040} \le S \le 0.63194 + \frac{1}{5040}$$

Example 2) Determine the number of terms required to approximate the sum of the series, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ with an error of less than three decimal places.

0.0005

$$|a_{n+1}| = \left|(-1)^{n+2}\frac{1}{(n+1)^3}\right| \le 0.0005.$$

$$\frac{1}{(n+1)^3} \le 0.0005 \Rightarrow (n+1)^3 \ge \frac{1}{0.0005}$$

$$(n+1)^3 \ge 2000$$

$$n = 12$$

Practice) **Approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$ for its first six terms.**

$$\sum_{n=1}^{6}(-1)^{n+1}\frac{n}{2^n} = \frac{1}{2} - \frac{2}{2^2} + \frac{3}{2^3} - \frac{4}{2^4} + \frac{5}{2^5} - \frac{6}{2^6} = \frac{3}{16}$$

$$|a_{n+1}| = \frac{7}{2^7}$$

$$\frac{3}{16} - \frac{7}{2^7} \le S \le \frac{3}{16} + \frac{7}{2^7}$$

$$\frac{17}{128} \le S \le \frac{31}{128}$$