IB Math 3: Alternating Series Name: _______________ Period: ________

Warm up

To prove the series' convergence, determine which test (Divergence test , Integral test, Direct comparison, Limit comparison, or Ratio test) you need to apply for the following series. Do not perform the test.

1. $\sum_{k=1}^{\infty} \frac{k}{e^k}$: Int. Test 2. $\sum_{k=1}^{\infty}\left(\frac{1}{3^k+9}\right)$: Direct comparison 3. $\sum_{k=1}^{\infty} k^{(-6/5)}$: Int. test 4. $\sum_{k=1}^{\infty}\left(\frac{(k!)^2}{(3k)!}\right)$: Ratio test.

$\lim_{a\to\infty}\int_1^a \frac{x}{e^x}dx$. $\sum \frac{1}{3^k+9} < \sum \frac{1}{3^k}$ $\sum \frac{1}{k^{6/5}}$. (p series)

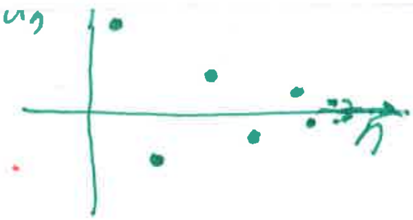
Part I: Alternating Series

When a series contains both positive and negative term alternating in sign, it is called alternating series. For example, $\sum_{k=0}^{\infty}(-\frac{1}{2})^k = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$......... is an alternating series. To test if the alternating series' convergence, the alternating series test is used. The following is the ALTERNATING SERIES TEST THEOREM.

Example 1) Test the series $\sum_{n=1}(-1)^{n+1}\frac{1}{n}$ for convergence.

Solution: $n=1$ $n=2$ $n=3$ $n=4$

$1 + -\frac{1}{2} + \frac{1}{3} + -\frac{1}{4}$. . . . .



Alt. Series test.

① $\lim_{n\to\infty}\frac{1}{n} = 0$

$\therefore \sum(-1)^{n+1}\frac{1}{n}$ converges.

② $a_n = \frac{1}{n} \Rightarrow \frac{d}{dn}\left(\frac{1}{n}\right) = \frac{d}{dn}(n^{-1}) = \frac{-1}{n^2} < 0$ for $n \in \mathbb{Z}^+$

Example 2) Test the series $\sum_{n=1}^{\infty}\frac{n}{(-2)^{n-1}}$ for convergence.

Solution: $\Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{2^{n-1}}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{2^n}\right) = a_n$.

Alt. Series test.

L'Hopital

① $\lim_{n\to\infty}\frac{2n}{2^n}\left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty}\frac{2}{\ln 2\cdot 2^n}\left(\frac{2}{\infty}\right) = 0$

② $a_n = \frac{2n}{2^n}$. $\Rightarrow \frac{d}{dn}\left(\frac{2n}{2^n}\right) = \frac{2(\cancel{2^n}) - 2n\cdot\ln 2\cdot\cancel{2^n}}{(2^n)^{\cancel{2}}}$

$= \frac{2-2n\cdot\ln 2}{2^n}$

$\therefore \sum\frac{n}{(-2)^{n-1}}$ converges.

$= \frac{2[1-n\cdot\ln 2]}{2^n} < 0$ for $n \in \mathbb{Z}^+$.

Part II: Absolute Convergence and Conditional Convergence for Alternating Series

Alternating series is classified differently as follows:

1. **Alternating Series,** $\sum_{n=1}^{\infty}(-1)^n a_n$ **, is absolutely convergent if** $\sum_{n=1}^{\infty}\left|(-1)^n a_n\right|$ **converges.**
2. **Alternating Series,** $\sum_{n=1}^{\infty}(-1)^n a_n$ **, is <u>conditionally convergent</u> if** $\sum_{n=1}^{\infty}(-1)^n a_n$ **converses but** $\sum_{n=1}^{\infty}\left|(-1)^n a_n\right|$ **diverges.**

Example) Determine whether each of the series is conditionally convergent or absolutely convergent.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$

I. $\sum\left|(-1)^n \cdot \frac{1}{\sqrt{n}}\right| = \sum \frac{1}{\sqrt{n}}$.

Int. test

$\lim_{a\to\infty}\int_1^a x^{-\frac{1}{2}}\,dx = \lim_{a\to\infty}\left(2\sqrt{a}-2\sqrt{1}\right) = \infty$.

diverges.

II. Alt. Series test.

$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ ✓ $\frac{d}{dn} n^{-\frac{1}{2}} = -\frac{1}{2} n^{-\frac{3}{2}} < 0$ for $n \in \mathbb{Z}^+$

$\therefore \sum(-1)^n \frac{1}{\sqrt{n}}$ converges.

$\Rightarrow$ conditionally converges.

b. $\sum_{n=1}^{\infty}\left(\frac{-1}{3}\right)^n$

I. $\sum\left(\frac{1}{3}\right)^n$

G. Series. $|r| = \left|\frac{1}{3}\right| < 1$

converges.

$\therefore \sum\left(-\frac{1}{3}\right)^n$ absolutely converges.