The Binomial Distribution

Bernoulli Trials: Trials of only two mutually exclusive possible outcomes.

$$\boxed{P(X=x)=\binom{n}{x}(p)^x(1-p)^{n-x}}$$

Where n is the number of trials, p is the probability of a success, and 1-p is the probability of a failure.

- $\boxed{X \sim B(n,p)}$: X is distributed binomially with parameters n and p.

Expected Value and Variance of Binomial Distribution

- $E(x)=np$
- $Var(x)=np(1-p)$

Ex) If $X \sim B(5,0.6)$, find $P(x=4)$.

$n=5$, $p=0.6$ $\Rightarrow$ $P(X=4)=\binom{5}{4}(0.6)^4(0.4)^{5-4}=0.2592.$

$=\dfrac{162}{625}$

=

Ex) A manufacture finds that 30% of the items produced from one of the assembly lines are defective. During a floor inspection, the manufacturer selects 10 items from this assembly line. Find the probability that the manufacturer finds.

a) **Two defectives**
b) **At least two defectives.**
c) **Find E(x).**
d) **Find the standard Deviation.**

{ Defective $p = 0.3$)

$n=10$

a) $P(X=2)=\binom{10}{2}(0.3)^2(0.7)^8.$

=

c) $E(x)=(10)(0.3)=3$

d) $\sigma=\sqrt{(10)(0.3)(0.7)}=\sqrt{2.1}$

≈ 1.45

b) $P(2\le X\le 10)=1-P(X=0)-P(X=1)$

$=1-\binom{10}{0}(0.3)^0(0.7)^{10}-\binom{10}{1}(0.3)^1(0.7)^9$

$=0.850692$

$\approx 0.851.$

Ex) The random variable X is such that $E(x)=8$ and $Var(x)=4.8$. Find $P(x=3)$.

(Binomial Distribution)

① $E(x)=n\cdot p=8$

② $Var(x)=n\cdot p\cdot(1-p)=4.8$

$(8)(1-p)=4.8$

$1-p=\frac{4.8}{8}$

$p=1-\frac{4.8}{8}=0.4.$

$8 = n(0.4) \Rightarrow n = \frac{8}{0.4} = \frac{80}{4} = 20$.

$p = 0.4$, $n = 20$ $\Rightarrow$ $P(X=3) = \binom{20}{3}(0.4)^3(0.6)^{20-3}$

$= 0.01235$

Poisson Distribution

The distribution of the number of events in a "random process".

Examples:

Random Process	Event
Telephone calls in a fixed time interval	Number of wrong calls in an hour (Time dependent)
Accident in a factory	Number of Accident in a day (Time dependent)
Flaws in a glass Panel	Number of flaws per square cm (Area dependent)
Bacteria in Milk	Number of bacteria per 2 liter (Volume dependent)

- **An event is as likely to occur in one given interval as it is in another.**
- **Events occur uniformly is proportional to the size of the time interval, area, or volume.**

The Poisson distribution formula:

$$P(X=x)=\frac{e^{-\mu}\mu^x}{x!} \text{ where } \mu=\lambda t \text{ and } x=0,1,2,3,\ldots\ldots$$

$$=\frac{e^{-m}\cdot m^x}{x!}$$

The Poisson Notation:

$X \sim Po(\lambda t)$: The random variable x has a poisson distribution with parameter λt.

$X \sim P_o(m)$ Where λ is a rate per unit and t is a time interval.

$m=\mu=\lambda t$

- X is the number of event in a time interval of length t with rate λ per unit time
-

Expected Value and Variance of the Poisson Distribution

$$E(x)=\mu \text{ and } Var(x)=\mu$$

Binomial Distribution.

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$E(X) = \mu = n \cdot p$

$\Rightarrow p\left(\frac{\mu}{n}\right)$

$$= \frac{n!}{(n-x)!\,x!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

$x \le n.$

$$= \frac{n(n-1)(n-2)\cdots(n-x)!}{(n-x)!\,x!} \left(\frac{\mu^x}{n^x}\right)\left(\frac{n-\mu}{n}\right)^n \left(\frac{n-\mu}{n}\right)^{-x}.$$

$$= \frac{n(n-1)(n-2)\cdots\cancel{(n-x)!}}{\cancel{(n-x)!}\,x!} \left(\frac{\mu^x}{\cancel{n^x}}\right)\left(1 - \frac{\mu}{n}\right)^n \left(\frac{\cancel{n^x}}{(n-\mu)^x}\right)$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))}{(n-\mu)^x} \cdot \frac{\mu^x}{x!} \left(1 - \frac{\mu}{n}\right)^n$$

Poisson Distribution.

$$\lim_{n\to\infty}\left[\frac{\overbrace{n(n-1)(n-2)\cdots(n-(x-1))}^{n^x \cdots}}{\underbrace{(n-\mu)^x}_{n^x}}\right]\left(1-\frac{\mu}{n}\right)^n \frac{\mu^x}{x!}$$

$\to 1 \qquad\qquad \to e^{-\mu}$

$$P(x) = \frac{e^{-\mu} \cdot \mu^x}{x!} = \frac{e^{-m} \cdot m^x}{x!}$$

$\mu = m$

2) Derive $E(x) = \mu$ for the Poisson Distribution.

$x = y + 1 \Rightarrow y = x - 1$

Binomial : $E(x) = n \cdot p$

$x = 1 \Rightarrow y = 0$

$x \to \infty \Rightarrow y \to \infty$

poisson: $\sum x \cdot p = \sum_{x=1}^{\infty} x \cdot \frac{e^{-\mu} \cdot \mu^x}{x!}$

$$= \sum_{y=0}^{\infty} \frac{(y+1) e^{-\mu} \mu^{y+1}}{(y+1)!\ y!} = \sum_{y=0}^{\infty} \frac{e^{-\mu} \cdot \mu \cdot \mu^y}{y!}$$

$$= e^{-\mu} \cdot \mu \cdot \sum_{y=0}^{\infty} \frac{\mu^y}{y!} = e^{-\mu} \cdot \mu\, e^{\mu} = \boxed{\mu}$$

Notes : $\sum_{y=0}^{\infty} \frac{\mu^y}{y!} = 1 + \frac{\mu}{1} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} \cdots$

$= e^{\mu}$ by M Series.

Ex) Faults occur on a piece of string at an average rate of one every three meters. Bobbins, each containing 5 meters of this string, are to be used. What is the probability that a randomly selected bobbin will contain.

a. Two faults.
b. At least two faults.

a) $P(X=2) = \dfrac{e^{-m} \cdot m^2}{2!} \Rightarrow m = (r)(t) = (\tfrac{1}{3})(5) = \tfrac{5}{3}$.

$= \dfrac{e^{-\frac{5}{3}} (\frac{5}{3})^2}{2!} \approx \boxed{0.262}$

$P(X \geq 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \dfrac{e^{-\frac{5}{3}} \cdot (\frac{5}{3})^0}{0!} - \dfrac{e^{-\frac{5}{3}} (\frac{5}{3})^1}{1!} \approx \cancel{0.500}$. 0.496

Ex) A radioactive source emits particles at an average rate of one every 12 seconds. Find the probability that at most 5 particles are emitted in one minute.

$m = (\frac{1}{12})(60) = \frac{10}{2} = \boxed{5}$.

$X \sim P_o(5) \Rightarrow P(0 \leq X \leq 5) = \dfrac{(e^{-5})(5)^0}{0!} + \left(\dfrac{e^{-5} \cdot 5^1}{1!}\right) + \dfrac{e^{-5} \cdot 5^2}{2!} + \dots + \dfrac{e^{-5} \cdot 5^5}{5!}$

≈ 0.616

Ex) A typist finds that they make two mistakes, on average, every three pages. Assuming that the number of errors per page follows a poisson distribution, what are the chances that there will be 2 mistakes in the nexst page they type?