The Beginnings of π

What is π?

The Greek letter π represents the mathematical constant whose value is the ratio of the ciumference (C) of a circle to its diameter (d):

$$\pi = C/d$$

It is an irrational number, which means that it cannot be expressed as a fraction, nor as a terminating or recurring decimal. It is also a *transcendental* number, which means it cannot be expressed as a finite sequence of algebraic operations (powers, roots, sums, etc) on integers. Its approximation to 50 decimal places is:

3.141 592 653 589 793 238 462 643 383 279 502 884 197 169 399 375 10

The Greek letter π was chosen by William Jones[1] in 1707 as it is the first letter of the Greek word for perimeter: "περίμετροζ".

Earliest evidence of the use of π

The earliest evidence of the use of an approximation for the value of π, 22/7, is in the designs of the pyramids in Egypt. The Great Pyramid, which was built around 2550-2500 BC, was precisely 1760 cubits[2] around the base with a height of 280 cubits. Taking the value of π as 22/7, we can see that the ratio of the perimeter of the base to the height is 2π, ie 1760/280 = 2*22/7. Similar proportions were used even earlier around 2600 BC for the Pyramid of Meidum.

Earliest written evidence of π

The concept of π and an approximation to its value was known to the Ancient Egyptians, Babylonians, Indians and Greeks. The earliest appearance of π in written text dates from around 1900 BC. For example, the Babylonians used the approximation 25/8 (= 3.125) and the Egyptians used 256/81 (= 3.16049…). In the Indian book *Shatapatha Brahmana*[3], π is approximated to 339/108 (= 3.138888…).

Archimedes (287-212 BC) was the first to estimate π rigorously. He calculated the perimeters of inner polygons inscribed in a circle of known radius and compared them with the perimeters of outer polygons (*see diagram*). In this way, he knew that the value of π must lie between these two estimates.

---

[1] Willaim Jones (1675-1749) was a Welsh mathematician who was a friend of Sir Isaac Newton and Sir Edmund Halley. He became a Fellow of the Royal Society (the national academy of Science in the UK) in 1712 and later became its vice-president.

[2] An ancient unit of linear measure, originally equal to the length of the forearm from the tip of the middle finger to the elbow, or about 17 to 22 inches (43 to 56 centimetres).

[3] A text in Hindu describing Vedic rituals.

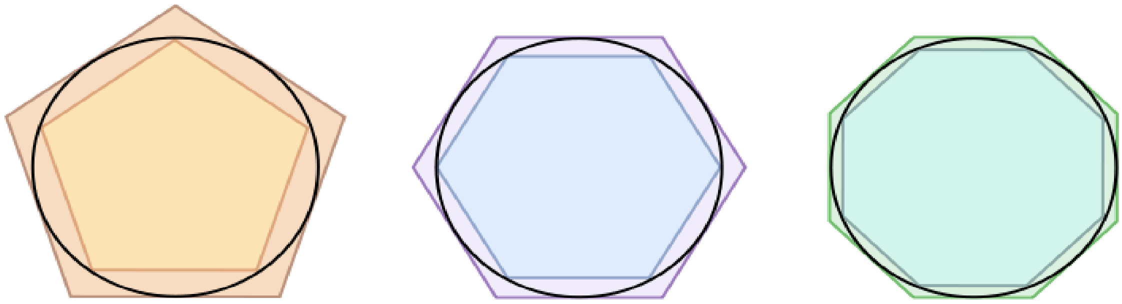

Figure 1: Archimedes' method of calculating π
Source: http://en.wikipedia.org/wiki/File:Archimedes_pi.svg

We can illustrate this method by constructing a spreadsheet. Letting the radius of the circle be 1 unit and taking n as the number of sides of the polygon, the perimeter of the outer polygon is:

$$P(\text{outer}) = n*2*\tan(360°/(2n°))$$

This is found by constructing isosceles triangles with their apexes at the centre of the circle and their bases as the sides of the polygon. Each triangle base will then be equal to $2*\tan(360°/(2n°))$. The total perimeter of the polygon is then n times the length of each side.

Unfortunately, the *Excel* spreadsheet package requires angles to be measured in radians and not degrees. The relationship between radians and degrees is:

$$360° = 2\pi \text{ radians}$$

So the half-angle at the apex of each triangle would be:

$$\theta = (360/2n)*(2\pi/360) = \pi/n \text{ radians}$$

So we appear to have come full circle. In reality, it is not possible to use a value of π when trying to find an estimate of it!

Also, it is worth noting that trigonometry was discovered by Hipparchus,[4]who lived from c.90 to c.120 BC, and later worked on by Ptolemy[5] (90–168) so Archimedes could not have used this method!

However I am going to go ahead with this method for the purpose of illustrating the process of converging estimates. As n tends towards infinity, the two different estimates will converge towards the same value, allowing us to calculate a value for π.

Employing the concepts of radians and trigonometry, the perimeter of the outer polygon is therefore:

---

[4] Hipparchus was a Greek scholar of the Hellenistic period.
[5] Ptolomy was a Roman citizen, of Greek or Egyptian ethnicity, living in Egypt under the Roman Empire.

P(outer) = n*2*tan(π/*n*)

And the perimeter of the inner circle is:
P(inner) = n*2*sin(π/*n*)

The spreadsheet then looks like this:

Table 1: Finding an approximate value for π

Number of sides	Perimeter of outer polygon	Perimeter of inner polygon	Average/diameter	π - (average/diameter)
4	8.000000000000000	5.656854249492380	3.414213562373090	-0.272620908783302
5	7.265425280053610	5.877852522924730	3.285819450744590	-0.144226797154792
6	6.928203230275510	6.000000000000000	3.232050807568880	-0.090458153979084
7	6.742044663305400	6.074372347645810	3.204104252737800	-0.062511599148011
9	6.551464216791640	6.156362579862040	3.176956699163420	-0.035364045573627
8	6.627416997969520	6.122934917841440	3.187587978952740	-0.045995325362946
9	6.551464216791640	6.156362579862040	3.176956699163420	-0.035364045573627
10	6.498393924658130	6.180339887498950	3.169683453039270	-0.028090799449475
11	6.459782844644070	6.198116250511450	3.164474773788880	-0.022882120199087
12	6.430780618346940	6.211657082460500	3.160609425201860	-0.019016771612067
13	6.408424438831410	6.222207271476500	3.157657927576980	-0.016065273987186
14	6.390817282924200	6.230586150776800	3.155350858425250	-0.013758204835457
15	6.376696850100660	6.237350724532780	3.153511893658360	-0.011919240068568
100	6.285253208670230	6.282151815625660	3.141851256073970	-0.000258602484179
200	6.283702129465970	6.282926924728270	3.141657263548560	-0.000064609958766
300	6.283414993378310	6.283070469747480	3.141621365781450	-0.000028712191654
400	6.283314503186870	6.283120710969070	3.141608803538980	-0.000016149949192
500	6.283267991889770	6.283143965558950	3.141602989362180	-0.000010335772388
1,000	6.283205978112310	6.283174971759130	3.141595237467860	-0.000002583878067
2,000	6.283190474897470	6.283182723323520	3.141593299555250	-0.000000645965453
10,000	6.283185513888110	6.283185203825330	3.141592679428360	-0.000000025838566
100,000	6.283185309246670	6.283185306146040	3.141592653848180	-0.000000000258385
10,000,000	6.283185307179790	6.283185307179480	3.141592653589820	-0.000000000000026
100,000,000	6.283185307179590	6.283185307179580	3.141592653589790	0.000000000000000

As the *Excel* value of π is correct to fifteen decimal places, all values in the table are given to fifteen decimal places also. The fourth column in the table is the average of the two perimeters divided by the diameter of the circles (equal to 2 units) which is our estimate of π.

The last column denotes the *Excel* value of π minus our estimate of π. We can see that this reaches zero when the polygons have 100,000,000 sides. Therefore our best estimate of π is 3.141 592 653 589 790 which, according to the value of π on the first page, is actually correct to fourteen decimal places, which is what we would expect.

Archimedes actually went as far as a 96-sided polygon and proved that π ≈ 3.1419.

Developments in China

In China, the ratio of the circumference of a circle to its diameter was often taken to be 3. However, Zhang Heng (78-139 AD) calculated that it as 3.1724 by considering the proportion of the celestial circle to the diameter of the earth (736/232), or as equal to $\sqrt{10}$.

Liu Hui was a Chinese mathematician who lived in the Wei Kingdom. In 263 AD he edited and published a commentary containing solutions to the mathematical problems contained in the famous Chinese book *The Nine Chapters on the Mathematical Art*. In particular, he noted that he was dissatified with previous estimates of π and set about finding new ways by looking at the areas of polygons inscribed in circles.

He stated that multiplying one side of an inscribed hexagon (6-sided polygon) by the radius and then multiplying it by 3 yielded the area of dodecagon (12-sided polygon). Then, if the product of one side of a dodecagon and the radius is multiplied by 6 the area of a 24-sided polygon is obtained, and so on until the resulting poygon becomes one with the circle and there is no difference. Thus, for polygons of n sides, the limit as $n \to \infty$ is equal to the area of the circle.

Given that the circle in question has a radius of 1 unit, this iterative process can be expressed as:

$$A_{12} = 3*m_6*1 = 3, \text{ since } m_6 = \text{radius} = 1$$

$$A_{24} = 2*3*m_{12}$$

$$A_{44} = 2*2*3*m_{24}$$

$$\ldots, \tag{1}$$

where A_n is the area of an n-sided polygon inscribed in a circle of radius 1 and m_n is the length of a side of an n-sided polygon.

Further, Liu Hui proved that:

area of a circle = ½ $C*R$

where C is the circumference of the circle and R is the radius.

To prove this, he stated that:

> "Between a polygon and a circle, there is excess radius. Multipy the excess radius with a side of the polygon. The resulting area exceeds the boundary of circle."

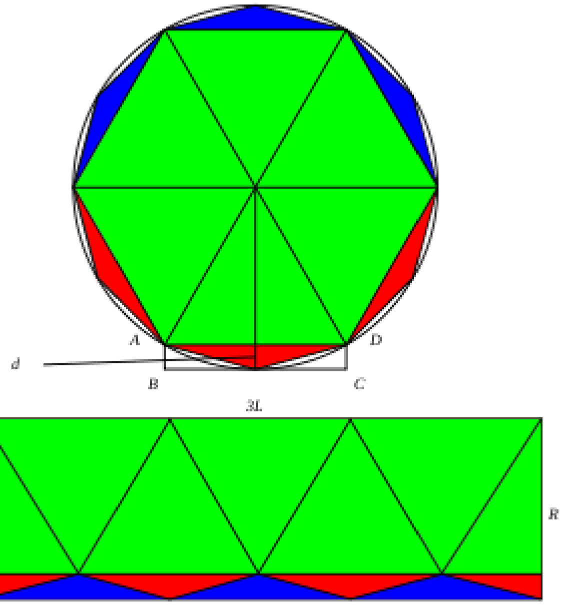

Figure 2: Liu Hui's method of calculating π

Source: <u>http://en.wikipedia.org/wiki/File:Cutcircle.svg</u>

In the circle in the above diagram, *d* is the excess radius. If we multiply *d* by the length of one side of the polygon we obtain an area equal to that of the rectangle *ABCD*. This rectangle clearly extends beyond the boundary of the circle. Thus, if the polygon has a very large number of sides, the excess radius becomes very small and so the area of the rectangle becomes even smaller. That is:

as $n \to \infty$, $d \to 0$ and the area of $ABCD \to 0$

Therefore, when $n \to \infty$, the perimeter of the *n*-sided polygon becomes closer and closer to the circumference of the circle. By comparing the circle with the rectangle in the diagram above, we can see that the area of the circle equals half the circumference multiplied by the radius.

Liu Hui's iterative algorithm

Liu Hui began with an inscribed hexagon.

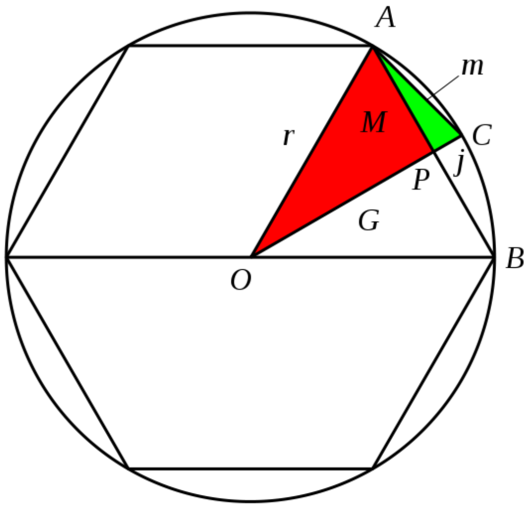

Figure 3: Liu Hui's iterative algorithm

Source: http://en.wikipedia.org/wiki/File:Liuhui_geyuanshu.svg

Let M be the length of AB, which is one side of the hexagon, and r the length of the radius of the circle. The line OPC bisects AB so that AC is one side of the dodecagon. Let the length of AC be m, the length of PC be j and the length of OP be G. The two triangles, AOP and APC are right angled.

Using the Gou Gu theorem[6],

$$G^2 = r^2 - [M/2]^2 \quad \text{so} \quad G = \sqrt{(r^2 - [M/2]^2)}$$

$$j = r - G = r - \sqrt{(r^2 - [M/2]^2)}$$

$$m^2 = [M/2]^2 + j^2 \quad \text{so} \quad m = \sqrt{([M/2]^2 + j^2)}$$

With a circle of radius 1 unit, we have:

$$\begin{aligned} m^2 &= (M/2)^2 + j^2 \\ &= M^2/4 + \{1 - \sqrt{(1^2 - [M/2]^2)}\}^2 \\ &= M^2/4 + \{1 - 2\sqrt{(1 - M^2/4)} + (1 - M^2/4\} \\ &= M^2/4 + 1 - 2\sqrt{(1 - M^2/4)} + 1 - M^2/4 \\ &= 2 - 2\sqrt{(1 - M^2/4)} \\ &= 2 - \sqrt{(4 - M^2)} \\ &= 2 - \sqrt{(2 + [2 - M^2])} \\ \text{So} \quad m &= \sqrt{\{2 - \sqrt{(2 + [2 - M^2])}\}} \end{aligned}$$

This provides with an iterative formula where m is the length of a side of the next polygon formed by bisecting the one before with side of length M.

Therefore, if the radius is 1 unit then $m_6 = 1$ and:

[6] The Gou Gu theorem is equivalent to Pythagoras' theorem and was known to Chinese mathematicians from around the 1st century AD.

$$m_{12} = \sqrt{\{2 - \sqrt{(2 + [2 - 1^2])}\}} = \sqrt{\{2 - \sqrt{(2 + 1)}\}}$$

$$m_{24} = \sqrt{\{2 - \sqrt{(2 + [2 - m_{12}{}^2])}\}} = \sqrt{\{2 - \sqrt{(2 + [2 - (2 - \sqrt{(2 + 1)})]}\}}$$
$$= \sqrt{\{2 - \sqrt{(2 + \sqrt{[2 + 1]})}\}}$$

Following this pattern, we get:

$$m_{48} = \sqrt{\{2 - \sqrt{\{2 + \sqrt{(2 + \sqrt{[2 + 1]})}\}}\}}$$

$$m_{96} = \sqrt{\{2 - \sqrt{\{2 + \sqrt{\{2 + \sqrt{(2 + \sqrt{[2 + 1]})}\}}\}}\}}$$

and so on.

Also, given the previous iterative method labelled (1) above, we have:

$$A_{12} = 3*m_6*1 = 3$$

$$A_{24} = 2*3*m_{12} = 6*\sqrt{\{2 - \sqrt{(2 + 1)}\}}$$

$$A_{44} = 2*2*3*m_{24} = 12*\sqrt{\{2 - \sqrt{(2 + \sqrt{[1 + 2]})}\}}$$

and so on.

We can illustrate this iterative process in a spreadsheet:

Liu Hui's iterative process for estimating π

n	***m*(*n*)**	***A*(*n*)**
6	1.0000000000	
12	0.5176380902	3.0000000000
24	0.2610523844	3.1058285412
48	0.1308062585	3.1326286133
96	0.0654381656	3.1393502030
192	0.0327234633	3.1410319509
384	0.0163622792	3.1414524723
768	0.0081812081	3.1415576079
1536	0.0040906126	3.1415838921
3072	0.0020453074	3.1415904632
6144	0.0010226538	3.1415921060
12288	0.0005113269	3.1415925166
24576	0.0002556635	3.1415926186
49152	0.0001278317	3.1415926453
98304	0.0000639159	3.1415926453
196608	0.0000319579	3.1415926453
393216	0.0000159790	3.1415926453
786432	0.0000079895	3.1415923038
1572864	0.0000039947	3.1415923038

In this spreadsheet, $M(n) = m_n$ and $A(n) = A_n$.

The area of a circle is ½C*r = πr^2 so if the radius is equal to 1 unit, the area is simply equal to π. Therefore in the table above the areas of the polygons A_n become closer and closer to π as n increases. In our table, the largest polygon has 1,572,864 sides giving an approximation of π equal to 3.1415923038, which is correct to seven decimal places.

With a more sophisticated (ie more accurate) spreadsheet package, it is possible to calculate π to 595 digits by calculating the area of a polygon with $6*2^{1000}$ sides.

Liu Hui's quicker method

Calculating the square roots of irrational numbers was a very tedious process in Liu Hui's time, so he worked on finding a short cut by comparing the areas of successive polygons. He found that the proportion of the difference in areas of successive polygons was approximately ¼.

To illustrate this method, let D_n denote the difference in areas of polygons with n sides. So if:

$$D_n = A_n - A_{n/2} \quad \text{and} \quad D_n \approx \tfrac{1}{4} D_{n/2}$$

then

$$\begin{aligned} A_n &= D_n + A_{n/2} \\ &\approx \tfrac{1}{4} D_{n/2} + A_{n/2} \end{aligned}$$

So, extending the above spreadsheet we obtain:

LiuHui's iterative processes for estimating π

n	***m*(*n*)**	***A*(*n*)**	***D*(*n*)**	**1/4*D*(*n*/2) + *A*(*n*/2)**
6	1.0000000000			
12	0.5176380902	3.0000000000		
24	0.2610523844	3.1058285412	0.1058285412	
48	0.1308062585	3.1326286133	0.0268000721	3.1322856765
96	0.0654381656	3.1393502030	0.0067215898	3.1393286313
192	0.0327234633	3.1410319509	0.0016817478	3.1410306005
384	0.0163622792	3.1414524723	0.0004205214	3.1414523879
768	0.0081812081	3.1415576079	0.0001051356	3.1415576026
1536	0.0040906126	3.1415838921	0.0000262842	3.1415838918
3072	0.0020453074	3.1415904632	0.0000065711	3.1415904632
6144	0.0010226538	3.1415921060	0.0000016428	3.1415921060
12288	0.0005113269	3.1415925166	0.0000004105	3.1415925167
24576	0.0002556635	3.1415926186	0.0000001021	3.1415926192
49152	0.0001278317	3.1415926453	0.0000000267	3.1415926442
98304	0.0000639159	3.1415926453	0.0000000000	3.1415926520
196608	0.0000319579	3.1415926453	0.0000000000	3.1415926453
393216	0.0000159790	3.1415926453	0.0000000000	3.1415926453
786432	0.0000079895	3.1415923038	-0.0000003415	3.1415926453
1572864	0.0000039947	3.1415923038	0.0000000000	3.1415922184

where $D(n) = A(n) - A(n/2)$ and the last column approaches the area of a circle with radius equal to 1 unit.
Therefore the last column provides us with estimates of π using this quicker method and we can see that the best approximation is accurate to six decimal places which is very close to the previous estimate using his Liu Hui's longer method.

Significance of Liu Hui's work

Liu Hui's algorithm was one of his most important contributions to ancient Chinese mathematics. From Liu's work Zu Chonzhi[7] (429-500), a prominent Chiene mathematician and astronomer during the Liu Song and Southern Qi Dynasties, calculated two approximations to π which remained the most accurate estimates for nearly one thousand years. These two values were 3.1415926 and 3.1415927.

He obtained this result by working with a polygon with 12,228 sides. This was very impressive as he only used "counting rods" to make his calculations. These were wooden sticks which needed to be laid out in certain patterns.

What this research means to me

I have really enjoyed researching into the beginnings of π as I had no idea that Chinese mathematics played such a bit part, nor did I realize that Chinese thinking was so advanced.

In particular, I found it very exciting to work on the same concepts as these brilliant mathematicians did centuries ago and, although the concepts were not that difficult to understand, I admire them for their original ways of thinking. I also found it very satisfying to construct the spreadsheets myself and to actually see these approximations coming to life.

This experience has given me the opportunity to put into historical perspective the contributions made by (albeit) a very small number of Chinese mathematicians, previously unknown to me. So now I find I have been given the necessary incentives to want to know more about the development of mathematics in China.

---

[7] Most of Zu Chonzhi's great mathematical works were recorded in a book called *Zhui Shu* which he wrote together with his son Zu Gengzhi. However, this book has not survived as it was lost during the Song Dynasty.

Bibliography

A History of Mathematics: Victor Katz. Harper Collins College Publishers; USA 1993.

Notices of the AMS; August 2002: *Learning from Liu Hui? A Different Way to Do Mathematics,* Christopher Cullen.

http://library.thinkquest.org/23062/pi.html : 15 July 2009

http://en.wikipedia.org/wiki/Pi: 27 July 2009

http://en.wikipedia.org/wiki/William_Jones_(mathematician): 29 July 2009

http://en.wikipedia.org/wiki/Liu_Hui’s_%CF%80_algorithm: 29 July 2009

http://royalsociety.org/: 29 July 2009

http://www.answers.com/topic/cubit; 29 July 2009

http://en.wikipedia.org/wiki/Talk:Pi/Archive_5: 30 July 2009

http://en.wikipedia.org/wiki/Zu_Chongzhi: 1 August 2009