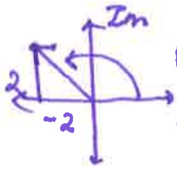
#1. $z^3 = -2+2i$




$\theta = \frac{3\pi}{4}$.

$r = \sqrt{2^2+2^2} = \sqrt{8}$

a) $z = (-2+2i)^{\frac{1}{3}}$

$z = \left(\sqrt{8}\,\text{Cis}\left(\frac{3\pi}{4}+2n\pi\right)\right)^{\frac{1}{3}}$

$z = 2^{\frac{3}{2}\cdot\frac{1}{3}}\,\text{Cis}\,\frac{1}{3}\left(\frac{3\pi}{4}+2n\pi\right)$

$n=0 \quad z = \sqrt{2}\,\text{CiS}\left(\frac{\pi}{4}\right) \quad \Rightarrow \quad \sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$

$n=1 \quad z = \sqrt{2}\,\text{Cis}\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)$

$n=2 \quad z = \sqrt{2}\,\text{Cis}\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)$

b) When $n=0$ $\quad z = \sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right] = \sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \boxed{1+i}$

#2. $z = x+iy$ , $|z| = \sqrt{x^2+y^2}$

$\Rightarrow \sqrt{x^2+y^2}+x+\boxed{yi} = 6\boxed{-2i} \quad \Rightarrow \quad \boxed{y=-2}$

$\sqrt{x^2+4}+x = 6$

$\left(\sqrt{x^2+4}\right)^2 = (6-x)^2$

$\cancel{x^2}+4 = 36-12x+\cancel{x^2}$

$12x = 32$

$x = \frac{32}{12} = \boxed{\frac{8}{3}}$

#3. $\dfrac{z}{z+2} = 2-i$

$z = (2-i)(z+2)$

$z = 2z + 4 - zi - 2i$

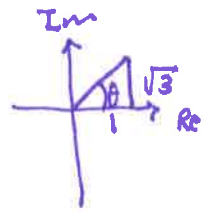
$zi - z = 4 - 2i$

$z(-1+i) = 4 - 2i$

$z = \dfrac{(4-2i)(-1-i)}{(-1+i)(-1-i)} = \dfrac{-4-4i+2i-2}{2} = \dfrac{-6-2i}{2} = \boxed{-3-i}$

#4. $z_1 = a + a\sqrt{3}i = a(1+\sqrt{3}i)$

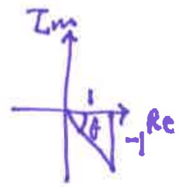
$= 2a\,\text{Cis}\left(\frac{\pi}{3}\right)$




$\theta = \dfrac{\pi}{3}$

$r = \sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}a$

$z_2 = 1 - i = \sqrt{2}\,\text{Cis}\left(-\frac{\pi}{4}\right)$




$\theta = -\dfrac{\pi}{4}$

$r = \sqrt{2}$

$\Rightarrow \left(\dfrac{z_1}{z_2}\right)^6 = \left(\dfrac{2a\,\text{Cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\,\text{Cis}\left(-\frac{\pi}{4}\right)}\right)^6 = \left(\frac{2a}{\sqrt{2}}\text{Cis}\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)^6$

$= (\sqrt{2}a)^6\,\text{Cis}\left[6\left(\frac{7\pi}{12}\right)\right]$

$= 8a^3\,\text{Cis}\left[\frac{7\pi}{2}\right]$

$= 8a^3\left[\cos\left(\frac{7\pi}{2}\right) + i\sin\left(\frac{7\pi}{2}\right)\right]$

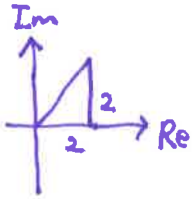
$= 8a^3\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$

$= 8a^3\left[0 + i(-1)\right] = \boxed{-8a^3 i}$

#5 . a) $W = 2+2i$

$|W| = \sqrt{2^2+2^2} = 2\sqrt{2}$

$Arg(w) = \frac{\pi}{4}$




b) $z = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = Cis\left(\frac{5\pi}{6}\right) \Rightarrow z^6 = Cis\, 6\left(\frac{5\pi}{6}\right)$

$= Cis(5\pi)$

$W = 2\sqrt{2}\, Cis\left(\frac{\pi}{4}\right) \Rightarrow W^4 = \left(\sqrt{8}\right)^4 \left(Cis\left(4\left(\frac{\pi}{4}\right)\right)\right)$

$= 2^{\frac{3}{2}\cdot \cancel{4}^{2}} Cis(\pi) = 2^6 Cis(\pi)$

$\Rightarrow W^4 \cdot z^6 = 2^6 Cis(\pi) \cdot Cis(5\pi) = 2^6\, Cis(6\pi) =$

$= 2^6\left[\cos 6\pi + i \sin 6\pi\right] = 2^6 = \boxed{64}$

#6. $u = 2+3i \quad v = 3+2i$

(a) $\frac{1}{u} + \frac{1}{v} = \frac{10}{W}$

$\frac{1}{u} = \left(\frac{1}{2+3i}\right)\frac{(2-3i)}{(2-3i)} = \frac{2-3i}{4+9} = \frac{2-3i}{13}$

$\frac{1}{v} = \frac{1}{(3+2i)}\frac{(3-2i)}{(3-2i)} = \frac{(3-2i)}{4+9} = \frac{3-2i}{13}.$

$\frac{1}{u} + \frac{1}{v} \Rightarrow \frac{2-3i}{13} + \frac{3-2i}{13} = \frac{5-5i}{13} = \frac{10}{W} \Rightarrow W = \frac{130}{5-5i} = \frac{26}{1-i} = \frac{26(1+i)}{2}$

$= 13+13i$

#7

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(a)

(i) $(\cos\theta + i\sin\theta)^5$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i\,10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$\Rightarrow \boxed{[\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta] + i\,[5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta]}$$

Equality.

(ii) De Moivre's theorem

$$(\cos\theta + i\sin\theta)^5 = (1)^5\,[\text{Cis}\,5(\theta + 2n\pi)]$$

$$= \boxed{\cos(5\theta + 10n\pi) + i\sin(5\theta + 10n\pi)}$$

(iii) $\therefore \cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\therefore \sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

(b) $z = r(\cos\delta + i\sin\delta) = r\,\text{Cis}\,\delta$.

$z^5 - 1 = 0 \Rightarrow z = (1)^{\frac{1}{5}}$ $\qquad 1 = \text{Cis}(360n)$

$= [\text{Cis}(360n)]^{\frac{1}{5}}$

$= [\text{Cis}\,\frac{1}{5}(360n)] = \text{Cis}(72n)$ where $n \in \mathbb{Z}$

$$\boxed{\delta = 72n \qquad r = 1}$$

(c) $16\sin^4\theta - 20\sin^2\theta + 5 = ?$

a(i) $\Rightarrow \sin 5\theta = 5\cos^4\theta\,\sin\theta - 10\cos^2\theta\,\sin^3\theta + \sin^5\theta$

$= \sin\theta\left[5(1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^3\theta + \sin^4\theta\right]$

$= \sin\theta\left[5(1-2\sin^2\theta+\sin^4\theta) - 10\sin^3\theta + 10\sin^5\theta + \sin^4\theta\right]$

$\sin(5\theta) = \sin\theta\left[16\sin^4\theta - 20\sin^2\theta + 5\right]$

when $\theta = 72^\circ \Rightarrow \sin(5\times 72^\circ) = \sin 360^\circ = 0$ where $\sin 72^\circ \neq 0$.

$\therefore 0 = \left[16\sin^4\theta - 20\sin^2\theta + 5\right]$

(d) $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

Q.F $\Rightarrow \sin^2\theta = \dfrac{20 \pm \sqrt{400-(4)(16)(5)}}{2\cdot 16} = \dfrac{20 \pm \sqrt{80}}{32} = \dfrac{5\pm\sqrt{5}}{8}$.

$\sin\theta = \cancel{\pm}\sqrt{\dfrac{5\pm\sqrt{5}}{8}}$

when $\theta = 72^\circ \Rightarrow \therefore \sin\theta = \sqrt{\dfrac{5\pm\sqrt{5}}{8}} = \dfrac{\sqrt{5\pm\sqrt{5}}\cdot\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}}$

$= \dfrac{\sqrt{10\pm 2\sqrt{5}}}{4}$

$\boxed{a = 10,\quad b = 2,\quad c = 5,\ \text{and}\ d = 4}$

#8.

(a) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\sin\theta\, i - 3\cos\theta\sin^2\theta - i\sin^3\theta$

By Binomial theorem.

Equality.

$(\cos\theta + i\sin\theta)^3 = (\text{Cis}\,\theta)^3 = \text{Cis}\,3\theta. = \cos 3\theta + i\sin 3\theta$

By De Moivre's theorem.

By Equality of Real Number

(b) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$\therefore \boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta}$

(c) $(\cos\theta + i\sin\theta)^5 \Rightarrow$ Binomial theorem

$(\cos\theta + i\sin\theta)^5 \Rightarrow$ De Moivre's theorem

Set Equal.

Using #7 (a) (iii)

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$= \boxed{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$

d) $\underbrace{\cos 5\theta}_{\text{from (c)}} + \underbrace{\cos 3\theta}_{\text{from (b)}} + \cos\theta = 0$

$\Rightarrow 16\cos^5\theta + 4\cos^3\theta - 20\cos^3\theta + 3\cos\theta = 0$

$= 16\cos^5\theta - 16\cos^3\theta + 3\cos\theta = 0$

$= \cos\theta\,[16\cos^4\theta - 16\cos^2\theta + 3] = 0$

$= \cos\theta\,[(4\cos^2\theta - 1)(4\cos^2\theta - 3)] = 0$ $\qquad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos\theta > 0$

$\cos\theta = 0 \qquad \cos\theta = \frac{1}{2}. \qquad \cos\theta = \frac{\sqrt{3}}{2}$

$$\boxed{\theta = \frac{\pi}{2}, -\frac{\pi}{2} \qquad \theta = \frac{\pi}{3}, -\frac{\pi}{3}, \qquad \theta = \frac{\pi}{6}, -\frac{\pi}{6}.}$$

e) $\cos 5\theta = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$\theta = \frac{\pi}{10} \qquad \cos\theta > 0$

$5\theta = \frac{\pi}{2} \quad (\cos 5\theta = 0 \text{ where } \theta = \frac{\pi}{10})$

$\cos(\frac{\pi}{10}) \neq 0.$

$\frac{\cancel{\cos\theta}\,[16\cos^4\theta - 20\cos^2\theta + 5] = 0}{\cancel{\cos\theta}}$

$16\cos^4\theta - 20\cos^2\theta + 5 = 0$

Q.F $\Rightarrow \cos^2\theta = \dfrac{20 \pm \sqrt{(20)^2 - (4)(20)(5)}}{2\cdot 16}$ $\Leftarrow$ only positive.

$\cos\frac{\pi}{10} = \sqrt{\dfrac{20 \pm 4\sqrt{5}}{32}} = \sqrt{\dfrac{5+\sqrt{5}}{8}}$

$\Rightarrow \therefore \cos(\frac{7\pi}{2}) = 0$

$\cos(\frac{7\pi}{10}) = -\sqrt{\frac{5+\sqrt{5}}{8}}$

(when $\theta = \frac{7\pi}{10} \Rightarrow 5\theta = \frac{7\pi}{2}$.)