Chapter 15 IB Problems Part 2

7. a. $O(0,0,0)$, $A(6,0,0)$, $B(6,-\sqrt{24},\sqrt{12})$, $C(0,-\sqrt{24},\sqrt{12})$

In OABC, the diagonals are $\overline{OB}$ and $\overline{AC}$.

$$\vec{OB} = \begin{pmatrix} 6 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} \qquad \vec{AC} = \begin{pmatrix} -6 \\ \sqrt{24} \\ -\sqrt{12} \end{pmatrix}$$

$\vec{OB}$ and $\vec{AC}$ have the same magnitude:

$$|\vec{OB}| = \sqrt{36+24+12} \qquad |\vec{AC}| = \sqrt{36+24+12}$$
$$= \sqrt{72} \qquad = \sqrt{72} \checkmark$$

$\vec{OB}$ and $\vec{AC}$ are perpendicular:

$$\vec{OB}\cdot\vec{AC} = (6)(-6) + (-\sqrt{24})(\sqrt{24}) + (\sqrt{12})(-\sqrt{12})$$
$$= 36 - 24 - 12 = 0 \checkmark$$

$\therefore$ OABC is a square.

b. $M\left(\frac{0+6}{2}, \frac{0-\sqrt{24}}{2}, \frac{0+\sqrt{12}}{2}\right)$

$M\left(3, \frac{-2\sqrt{6}}{2}, \frac{2\sqrt{3}}{2}\right) \rightarrow \boxed{M(3,-\sqrt{6},\sqrt{3})}$

c. $$\vec{OA}\times\vec{OB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = \begin{pmatrix} 0-0 \\ -(6\cdot\sqrt{12}-0) \\ 6(-\sqrt{24})-0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6\cdot 2\sqrt{3} \\ -6\cdot 2\sqrt{6} \end{pmatrix} = -12\sqrt{3}\begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

$\therefore \vec{n} = \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$ is normal to Π.

$$\begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y + \sqrt{2}z = 0$ is an equation for Π.

d. $L: \vec{r} = \begin{pmatrix} 3 \\ -\sqrt{6} \\ \sqrt{3} \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$

e. $y = 0 \rightarrow -\sqrt{6} + t = 0$ $\quad x = 3 + \sqrt{6} \cdot 0 = 3$

$t = \sqrt{6}$ $\quad z = \sqrt{3} + \sqrt{6} \cdot \sqrt{2} = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$

$D(3, 0, 3\sqrt{3})$

f. $t = -\sqrt{6} \rightarrow x = 3 + \sqrt{6} \cdot 0,\quad y = -\sqrt{6} - \sqrt{6} \cdot 1,\quad z = \sqrt{3} - \sqrt{6} \cdot \sqrt{2}$

$x = 3 \quad y = -2\sqrt{6} \quad = \sqrt{3} - 2\sqrt{3}$

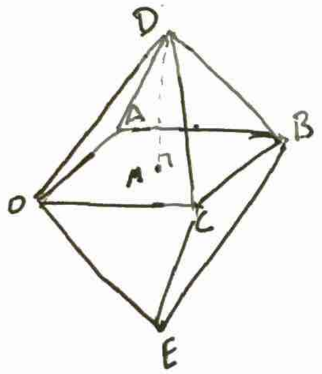
$= -\sqrt{3}$

$E(3, -2\sqrt{6}, -\sqrt{3})$

g i. $\overrightarrow{DO} = \begin{pmatrix} -3 \\ 0 \\ -3\sqrt{3} \end{pmatrix}$ $\quad \overrightarrow{DA} = \begin{pmatrix} 3 \\ 0 \\ -3\sqrt{3} \end{pmatrix}$ $\quad \cos\theta = \dfrac{-9 + 0 + 27}{\sqrt{9+6+27}\sqrt{9+6+27}} = \dfrac{18}{\sqrt{36}\sqrt{36}} = \dfrac{1}{2}$

$\theta \approx 60°$

ii.



OABCDE is a regular octahedron.

(Also acceptable: made of of 8 regular triangles)

8.

a. $\vec{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ $\quad L_1 : \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

b. $L_2 : \vec{r} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

L_1 and L_2 are not parallel since $\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \neq a \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ for any $a \in \mathbb{R}$.

$1 + t = 1 + 3s$ $\qquad 0 + 3t = -2 + s$ $\qquad 4 - 5t = -1 - 2s$

$t = 3s \longrightarrow 3(3s) = -2 + s$ $\qquad 4 - 5\left(\frac{-3}{4}\right) \overset{?}{=} -1 - 2\left(-\frac{1}{4}\right)$

$8s = -2$

$t = -\frac{3}{4} \longleftarrow s = -\frac{1}{4}$ $\qquad 4 + \frac{15}{4} \overset{?}{=} -1 + \frac{1}{2}$

$\frac{31}{4} \neq -\frac{1}{2}$

L_1 and L_2 do not intersect.

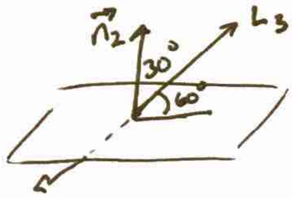
$\therefore$ L_1 and L_2 are skew.

c. $\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6+5 \\ -15+2 \\ 1-9 \end{pmatrix} = \begin{pmatrix} -1 \\ -13 \\ -8 \end{pmatrix} = \vec{n}$

$\begin{pmatrix} -1 \\ -13 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -13 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \rightarrow -x - 13y - 8z = 0 - 13 + 16$

$\boxed{-x - 13y - 8z = 3}$

di. The angle between $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix}$ is $30°$



$\cos 30° = \frac{k + 1 + 0}{\sqrt{1+1}\sqrt{k^2+1+1}}$ $\qquad 0 = \frac{1}{2}k^2 - 2k + 2$

$\frac{\sqrt{3}}{2} = \frac{k+1}{\sqrt{2}\sqrt{k^2+2}}$ $\qquad 0 = k^2 - 4k + 4$

$\frac{\sqrt{6}}{2} \cdot \sqrt{k^2+2} = k + 1$ $\qquad 0 = (k-2)^2$

$\frac{6}{4}(k^2+2) = k^2 + 2k + 1$ $\qquad \boxed{k = 2}$

$\frac{3}{2}k^2 + 3 = k^2 + 2k + 1$

8 d ii. $L_3 : \begin{cases} x = 3 + 2\lambda \\ y = \lambda \\ z = 1 - \lambda \end{cases}$ $\quad \Pi_2 : x + y = 12$

$3 + 2\lambda + \lambda = 12$

$3\lambda = 9$

$\lambda = 3$

$P(3+6, 3, 1-3)$

$\boxed{P(9, 3, -2)}$

9.

$l_1 : \vec{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $l_2 : \vec{r}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

a. $\vec{PQ} = \begin{pmatrix} 1 + \mu - \lambda \\ -1 - \mu - \lambda \\ -4 + 2\mu - \lambda \end{pmatrix}$

b. $\lambda = -\frac{6}{7}, \ \mu = \frac{5}{7}$

(see Part 1 #2)

c. when $\lambda = -\frac{6}{7}, \mu = \frac{5}{7}$

$$|\vec{PQ}| = \left| \begin{pmatrix} 1 + \frac{5}{7} + \frac{6}{7} \\ -1 - \frac{5}{7} + \frac{6}{7} \\ -4 + \frac{10}{7} + \frac{6}{7} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{18}{7} \\ -\frac{6}{7} \\ -\frac{12}{7} \end{pmatrix} \right| = \frac{6}{7} \left| \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \right| = \frac{6}{7}\sqrt{9+1+4}$$

$$= \boxed{\frac{6}{7}\sqrt{14}} \approx 3.21$$

d. $\vec{n} = \vec{v} \times \vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+1 \\ 1-2 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

$(1,2,3)$ is on l_1 $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

$3x - y - 2z = 3 - 2 - 6$

$\boxed{3x - y - 2z = -5}$

9e. $\vec{OT} = \vec{OB} + \eta(\vec{v} \times \vec{w})$

$\Pi : 3x - y - 2z = -5$

$$\vec{OT} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \eta \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$3(2+3\eta) - (1-\eta) - 2(-1-2\eta) = -5$$

$$6 + 9\eta - 1 + \eta + 2 + 4\eta = -5$$

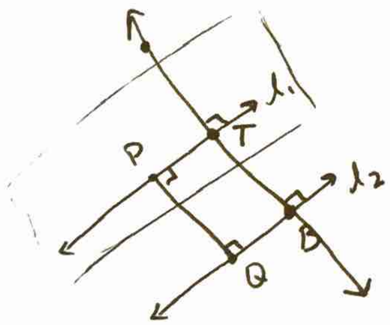
$$14\eta = -12$$

$$\boxed{\eta = -\frac{6}{7}}$$

f. $\vec{BT} = \vec{OT} - \vec{OB}$

$$\vec{BT} = \eta(\vec{v} \times \vec{w}) = -\frac{6}{7}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$|\vec{BT}| = \frac{6}{7}\sqrt{9+1+4} = \boxed{\frac{6}{7}\sqrt{14}}$$



g. The answers to c and f are the same.

$\vec{BT}$ is perpendicular to l_1 and l_2 and hence is perpendicular to Π.

So $|\vec{BT}|$ is the distance from l_2 to Π and the distance from l_1 to l_2.