Chapter 15 IB Questions Part 1

1. $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ $L_1: \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

2. $l_1: \vec{r_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $l_2: \vec{r_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

a. If P is on l_1, then $\overrightarrow{OP} = \begin{pmatrix} 1+\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix}$

If Q is on l_2, then $\overrightarrow{OQ} = \begin{pmatrix} 2+\mu \\ 1-\mu \\ -1+2\mu \end{pmatrix}$

So $\overrightarrow{PQ} = \begin{pmatrix} 2+\mu \\ 1-\mu \\ -1+2\mu \end{pmatrix} - \begin{pmatrix} 1+\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} 1+\mu-\lambda \\ -1-\mu-\lambda \\ -4+2\mu-\lambda \end{pmatrix}$

b. If $\overrightarrow{PQ} \perp l_1$, then $\overrightarrow{PQ} \cdot \vec{v} = 0$
and $\overrightarrow{PQ} \perp l_2$ $\overrightarrow{PQ} \cdot \vec{w} = 0$

$(1+\mu-\lambda)(1) + (-1-\mu-\lambda)(1) + (-4+2\mu-\lambda)(1) = 0$

$-4 + 2\mu - 3\lambda = 0 \rightarrow 2\mu = 3\lambda + 4$

$(1+\mu-\lambda)(1) + (-1-\mu-\lambda)(-1) + (-4+2\mu-\lambda)(2) = 0$

$-6 + 6\mu - 2\lambda = 0$

$-6 + 3(3\lambda+4) - 2\lambda = 0$

$7\lambda = 6$

$\boxed{\lambda = \frac{6}{7}}$

$2\mu = 3\left(\frac{6}{7}\right) + 4$

$\boxed{\mu = \frac{23}{7}}$ $\frac{5}{7}$

3. $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ $\quad \vec{w} = \begin{pmatrix} 4 \\ \lambda \\ 10 \end{pmatrix}$

a. $\vec{v}$ & $\vec{w}$ parallel $\Rightarrow$ $\vec{w} = 2\vec{v}$

$\lambda = 2(3)$

$\boxed{\lambda = 6}$

b. $\vec{v}$ & $\vec{w}$ perpendicular $\Rightarrow$ $\vec{w} \cdot \vec{v} = 0$

$8 + 3\lambda + 50 = 0$

$3\lambda = -58$

$\boxed{\lambda = -\frac{58}{3}}$

c. $\cos 10° = \dfrac{8 + 3\lambda + 50}{\sqrt{4+9+25}\,\sqrt{16+\lambda^2+100}}$

$\sqrt{38}\cos 10° \sqrt{\lambda^2+116} = 58 + 3\lambda$

$\boxed{\lambda \approx 3.73,\ 8.76}$ (GFC)

4.
a. $\vec{AB} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ $\quad \vec{AC} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

$\vec{AB} \times \vec{AC} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+6 \\ 2-0 \\ 0+2 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}}$

b. Area $= \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}\sqrt{16+4+4}$

$= \frac{1}{2}\sqrt{24}$

$= \frac{1}{2} \cdot 2\sqrt{6} = \boxed{\sqrt{6}}$

5. a. $\vec{PQ} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ $\quad L_1 : \vec{r} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$

b. $L_2 : \vec{r} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$

$-4 + 5s = -3 + 8t$ $\quad 0 + 2s = -1 + 6t$ $\quad 4 + 0s = 2 + 4t$

$2s = -1 + 6\left(\frac{1}{2}\right)$ $\quad 4 = 2 + 4t$

$2s = 2$ $\quad 2 = 4t$

$s = 1$ $\quad \frac{1}{2} = t$

If $s = 1$, then point on L_2 is $(-4 + 1\cdot 5,\ 0 + 1\cdot 2,\ 4 + 1\cdot 0)$

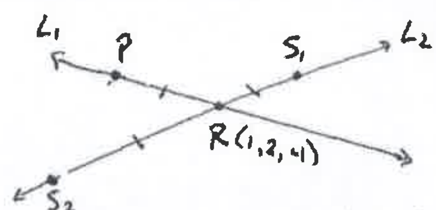
$(1, 2, 4)$ ✓

If $t = \frac{1}{2}$, then point on L_1 is $(-3 + \frac{1}{2}\cdot 8,\ -1 + \frac{1}{2}\cdot 6,\ 2 + \frac{1}{2}\cdot 4)$

$(1, 2, 4)$ ✓

c. $\cos\theta = \dfrac{5\cdot 8 + 2\cdot 6 + 0\cdot 4}{\sqrt{116}\ \sqrt{29}} = \dfrac{52}{\sqrt{116}\sqrt{29}}$

$\theta \approx 26.3°$



d. $\vec{RP} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}$

$\vec{RS} = \underbrace{\begin{pmatrix} -4 + 5s \\ 2s \\ 4 \end{pmatrix}}_{\vec{OS}} - \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{\vec{OR}} = \begin{pmatrix} -5 + 5s \\ 2s - 2 \\ 0 \end{pmatrix}$

$s = 0 \rightarrow \vec{OS} = \begin{pmatrix} -4 + 0 \\ 2\cdot 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix}$ ✓

$s = \frac{28}{29} \rightarrow \vec{OS} = \begin{pmatrix} -4 + \frac{28}{29}(5) \\ 2\left(\frac{28}{29}\right) \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{24}{29} \\ \frac{56}{29} \\ 4 \end{pmatrix}$

$|\vec{RP}| = |\vec{RS}|$

$\sqrt{16 + 9 + 4} = \sqrt{(5s - 5)^2 + (2s - 2)^2 + 0^2}$

$29 = 25s^2 - 50s + 25 + 4s^2 - 8s + 4$

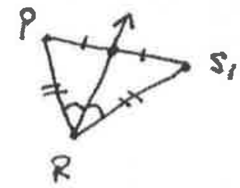
$0 = 29s^2 - 58s$

$0 = s(29s - 58)$

$s = 0,\ 2$

$\boxed{\begin{array}{l} S_1 (-4, 0, 4) \\ S_2 (6, 4, 4) \end{array}}$

5 e. Since $|\overrightarrow{PR}| = |\overrightarrow{RS_1}|$, then the midpoint of $\overline{PS_1}$ is on the perpendicular bisector of $P\hat{R}S_1$.

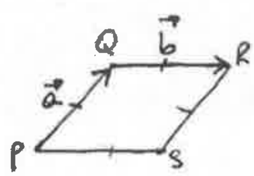


The midpoint is $M\left(\frac{-3+(-4)}{2}, \frac{-1+0}{2}, \frac{2+4}{2}\right)$

$$M\left(-\frac{7}{2}, -\frac{1}{2}, 3\right)$$

$$\overrightarrow{RM} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{1}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} \\ -\frac{5}{2} \\ -1 \end{pmatrix} \quad \boxed{\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + k\begin{pmatrix} -\frac{9}{2} \\ -\frac{5}{2} \\ -1 \end{pmatrix}}$$

6.



a. $\overrightarrow{PR} = \vec{a} + \vec{b}$

$\overrightarrow{QS} = \vec{b} - \vec{a}$

b. $\overrightarrow{PR} \cdot \overrightarrow{QS} = (\vec{a} + \vec{b}) \cdot (\vec{b} - \vec{a})$

$= \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a}$

$= \vec{a}\cdot\vec{b} - |\vec{a}|^2 + |\vec{b}|^2 - \vec{a}\cdot\vec{b}$

$= -|\vec{a}|^2 + |\vec{b}|^2$

$= 0$ since $|\vec{a}| = |\vec{b}|$ (PQRS is a rhombus)

Hence $\overrightarrow{PS}$ and $\overrightarrow{QS}$, the diagonals of the rhombus, are perpendicular.