Chapter 18 (Derivatives) ZB Questions solution.

#1. $8\ln x \cdot \frac{dy}{dx} + \frac{8y}{x} - 4x + 8y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{4x - \frac{8y}{x}}{8\ln x + 8y}$$

When $x=1$

$8y\cdot\cancel{\ln(1)}^{\,0} - 2 + 4y^2 = 7$

$4y^2 = 9 \qquad y = \pm\frac{3}{2}$

① $\left(x=1 \quad y=\frac{3}{2}\right)$

$$\frac{dy}{dx} = \frac{4 - \cancel{8}^{\,4}\left(\frac{3}{\cancel{2}}\right)}{0 + \cancel{8}_{\,4}\left(\frac{3}{\cancel{2}}\right)} = \frac{-8}{12} = \boxed{\frac{-2}{3}}$$

② $\left(x=1 \quad y=-\frac{3}{2}\right)$

$$\frac{dy}{dx} = \frac{4 + \cancel{8}^{\,4}\left(\frac{3}{\cancel{2}}\right)}{0 + \cancel{8}^{\,4}\left(-\frac{3}{\cancel{2}}\right)} = \frac{16}{-12} = \boxed{-\frac{4}{3}}$$

#2. $f(x) = [\ln(x-2)]^2$

$$\frac{df}{dx} = \frac{2\ln(x-2)}{x-2}.$$

$$\frac{d^2f}{dx^2} = \frac{\left(\frac{2}{\cancel{x-2}}\right)(\cancel{x-2}) - 2\ln(x-2)(1)}{(x-2)^2} = \boxed{\frac{2-2\ln(x-2)}{(x-2)^2}}$$

#3. $f(x) = \frac{\ln x}{x}$ $\qquad \frac{df}{dx} = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$

$$\frac{d^2f}{dx^2} = \frac{\left(\frac{-1}{\cancel{x}}\right)(x^{\cancel{2}}) - 2x(1-\ln x)}{x^4} = \frac{-x-2x(1-\ln x)}{\cancel{x^4}\,x^3} = \frac{-1-2(1-\ln x)}{x^3} = 0$$

continue ⇒

$\Rightarrow \quad -1-2(1-\ln x)=0$

$1-\ln x = \frac{-1}{2}$

$+\ln x = +\frac{3}{2}$

$\boxed{x = e^{3/2} = \sqrt{e^3}}$

#4. $f(x) = \frac{x^2}{e^x}$ $\Rightarrow$ Solve for x & y for $\frac{df}{dx} = 0$.

$\frac{df}{dx} = \frac{2x\,\cancel{e^x} - x^2 \cdot \cancel{e^x}}{(e^x)^{\cancel{2}}} = \frac{2x - x^2}{e^x} = 0 \quad \Rightarrow x[2-x]=0$

$\boxed{x=0 \quad x=2}$ for $x \geq 0$

$\Rightarrow \boxed{(0,0)\,;\ (2, \frac{4}{e^2})}$

#5. $f(x) = e^x - x^e \qquad x \geq 0$

(a) (i) $\boxed{f'(x) = e^x - e x^{e-1}}$

(ii) $e^x - e x^{e-1} = 0$

Solve by Guess and check

OR use a Graphing Cal.

$\boxed{x=1,\ x=e}$

#6. $f(x) = e^x \sin x$

(a) $f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

$f''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$= e^x \cancel{\sin x} + e^x \cos x + e^x \cos x - \cancel{e^x \sin x}$

$= 2e^x \cos x = \boxed{2e^x \sin(x + \frac{\pi}{2})}$

$\Rightarrow$ continue

#6 (b) $f''(x) = 2e^{x}\sin(x+\frac{\pi}{2})$

$$f^{3}(x) = 2e^{x}\sin(x+\tfrac{\pi}{2}) + 2e^{x}\cos(x+\tfrac{\pi}{2})$$

$$= 2e^{x}[\sin(x+\tfrac{\pi}{2}) + \cos(x+\tfrac{\pi}{2})]$$

$$f^{4}(x) = 2e^{x}[\cancel{\sin(x+\tfrac{\pi}{2})} + \cos(x+\tfrac{\pi}{2})] + 2e^{x}[\cos(x+\tfrac{\pi}{2}) - \cancel{\sin(x+\tfrac{\pi}{2})}]$$

$$= 4e^{x}\cos(x+\tfrac{\pi}{2}) = \boxed{-4e^{x}\sin x}$$

$$\cos(x+\tfrac{\pi}{2}) = -\sin x$$

#7. (a) $\lim_{x\to 2^-}(2x-1) = \lim_{x\to 2^+}(ax^2+bx-5)$

$$3 = 4a+2b-5 \Rightarrow 4a+2b = 8$$

$$\boxed{2a+b=4}$$

$$\lim_{x\to 2^+}(2x-1)' = \lim_{x\to 2^+}(ax^2+bx-5)'$$

$$\lim_{x\to 2^+} 2 = \lim_{x\to 2^-}(2ax+b)$$

$$\Rightarrow \boxed{2 = 4a+b}$$

$4a+b=2$

$2a+b=4$

$2a=-2$

$$\boxed{\begin{aligned} a &= -1 \\ b &= 6 \end{aligned}}$$

		Domain		Range
$f(x) =$	$2x-1$	$x \le 2$	$\Rightarrow$	$y \le 3$
	$-x^2+6x-5$	$2 < x < 3$	$\Rightarrow$	$3 < y < 4$

$-4+12-5 = 3$

$-9+18-5 = 4.$

$$x = 2y-1 \Rightarrow \boxed{y = \frac{x+1}{2} \quad \text{for} \quad x \le 3}$$

$$-y^2+6y-5 = x$$

$$y^2-6y+5 = -x$$

$$y^2-6y+9 = -x-5+9$$

$$(y-3)^2 = -x+4 \Rightarrow y = \cancel{\pm}\sqrt{4-x}+3 \quad \text{for } 3 < x < 4$$

$$\boxed{-\sqrt{4-x}+3}$$

$\Rightarrow$

$$\boxed{f^{-1} = \begin{cases} \frac{x+2}{2} & x \le 3 \\ -\sqrt{4-x}+3 & 3 < x < 4 \end{cases}}$$

#8. (a) $(x+h)^3 = x^3 + 3x^2h + 3h^2x + h^3$

(b) $\displaystyle \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3h^2x + h^3 \cancel{-x^3}}{h}$

$\displaystyle = \lim_{h\to 0} \frac{\cancel{h}[3x^2+3hx+h^2]}{\cancel{h}} = 3x^2$

#9. $y(x) = xe^{3x} \quad x \in R$

(a) $\frac{dy}{dx} = e^{3x} + 3xe^{3x}$

(b) Induction $\frac{d^ny}{dx^n} = n3^{n-1}e^{3x} + x3^ne^{3x}$ for $n \in Z^+$

1) When $n=1$

$\frac{dy}{dx} = e^{3x} + 3xe^{3x} = (1)3^{1-1}e^{3x} + x\cdot 3^1 \cdot e^{3x}$ $\Rightarrow$ The statement is true.

$= e^{3x} + 3xe^{3x}$

2) When $n=k \quad k \in Z^+$

Assume $\frac{d^ky}{dx^k} = k3^{k-1}e^{3x} + x3^ke^{3x}$ is true.

3) When $n = k+1$

$\frac{d^{k+1}y}{dx^{k+1}} = k\cdot 3^{k-1}\cdot 3e^{3x} + 3^ke^{3x} + x3^k\cdot 3e^{3x}$

$= k3^ke^{3x} + 3^ke^{3x} + x3^{k+1}e^{3x} = \boxed{(k+1)(3^{(k+1)-1}e^{3x}) + x3^{k+1}e^{3x}}$

$\Rightarrow$

The statement is true for $n=k+1$.

$$\therefore \quad \frac{d^ny}{dx^n} = n\,3^{n-1}e^{3x} + x\,3^n e^{3x} \text{ will be true for } n \in \mathbb{Z}^+$$

#10. $y_1 = |x+1|$

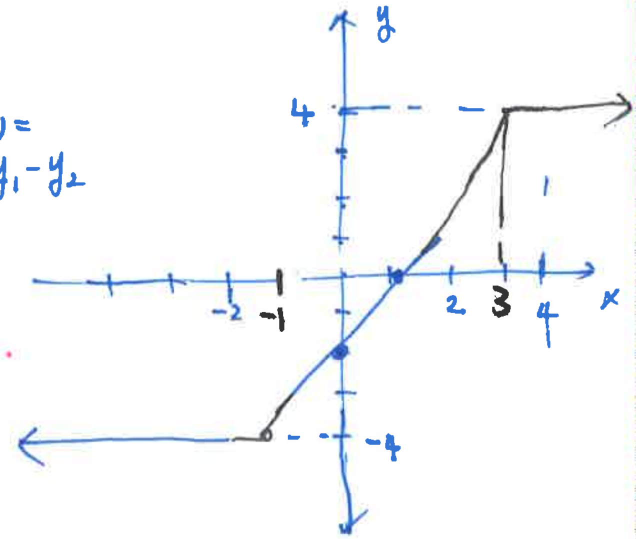
$y_2 = |x-3|$

x	y_1	y_2	$y_1 - y_2$
-4	3	7	-4
-2	1	5	-4
-1	0	4	-4
0	1	3	-2
2	3	1	2
3	4	0	4
4	5	1	4
6	7	3	4

$f(x) =$

$\Rightarrow y_1 - y_2$

$\Rightarrow$ plot.



(i) $f'(-3) = 0$

(ii) $f'(2.7) = \frac{8}{4} = 2$

#11. $f(x) = \left(\frac{x}{1-x}\right)^{\frac12}$

$$\frac{df}{dx} = \frac12\left(\frac{x}{1-x}\right)^{-\frac12}\left(\frac{(1-x) - x(-1)}{(1-x)^2}\right)$$

$$= \frac12\sqrt{\frac{1-x}{x}} \cdot \frac{1}{(1-x)^2}$$

$$= \frac12 \frac{(1-x)^{\frac12}}{x^{\frac12}} \cdot (1-x)^2 = \frac12 \frac{1}{(1-x)\sqrt{1-x}} = \frac{\sqrt{1-x}}{2(1-x)^2} = 0$$

$x \neq 1$

No solution.

$$\frac{d^2y}{dx^2} = \left(\frac12 \cdot x^{-\frac12} \cdot (1-x)^{-\frac32}\right)' = \frac12\left(-\frac12\right)x^{-\frac32}(1-x)^{-\frac32} + \left(\frac12\right)x^{-\frac12}\left(+\frac32\right)(1-x)^{-\frac52}$$

$$= -\frac14 x^{-\frac32}(1-x)^{-\frac52}\left[(1-x) - 3x\right] = 0$$

$$\Rightarrow 1 - 4x = 0 \qquad \boxed{x = \frac14}$$