Chapter 19 IB Questions Solution.

#1. (a) i) $\frac{df}{dx} = e^x - e \cdot x^{e-1}$ $(x \geq 0)$

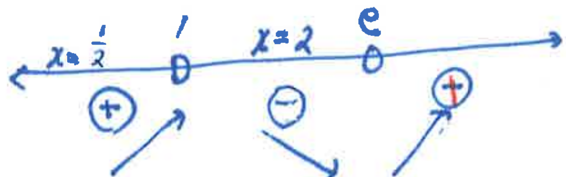
ii) $e^x - e x^{e-1} = 0$

$= e^x - \frac{e \cdot x^e}{x} = 0$

$\Rightarrow$ when $\boxed{x = 1}$ $\Rightarrow e^1 - \frac{e \cdot 1^e}{1} = 0.$ by observation.

when $\boxed{x = e}$ $\Rightarrow e^e - \frac{\cancel{e} \cdot e^e}{\cancel{e}} = 0.$

(b) Sign diagram of $f'(x)$



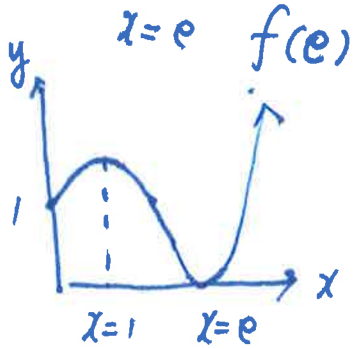
$f'(x) = e^x - \frac{e \cdot x^e}{x}$

$f'(\frac{1}{2}) = e^{\frac{1}{2}} - \frac{e \cdot (\frac{1}{2})^e}{\frac{1}{2}} = \sqrt{e} - 2 \cdot e \cdot 2^{-e} > 0$

$f'(2) = e^2 - e \cdot 2^{e-2} < 0$

$x = 1$ $f(1)$ is max.

$x = e$ $f(e)$ is min.



$f(x) = e^x - x^e$

$f(0) = 1$

$f(e) = e^e - e^e = 0$

(c) Since $f(x) = e^x - x^e > 0$ where $x > e$.

$\Rightarrow e^x > x^e$ $\pi > e$

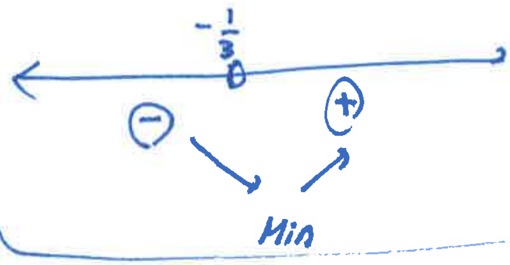
$\Rightarrow e^\pi > \pi^e.$

#2 $\quad y(x) = x \cdot e^{3x} \qquad x \in R.$

(c) $\dfrac{dy}{dx} = e^{3x} + 3x\,e^{3x} = 0.$

$e^{3x}(1+3x) = 0 \qquad x = -\frac{1}{3}.$ (critical point)

Sign diagram of $y' = e^{3x}(1+3x)$



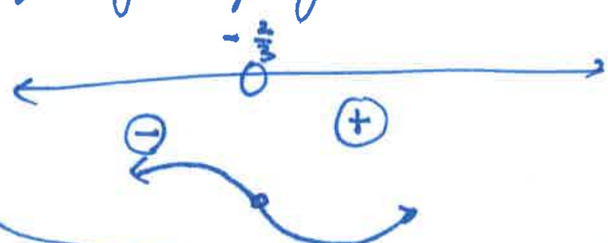
⇒ when $x = -\frac{1}{3}$ $f(-\frac{1}{3})$ is Local min. by the first derivative test.

(d) $\dfrac{dy}{dx} = e^{3x}(1+3x)$

$\dfrac{d^2y}{dx^2} = 3(e^{3x}) + 3(1+3x)e^{3x} = 0$

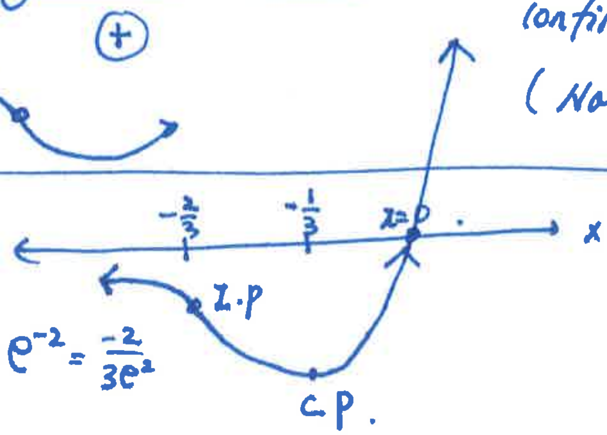
$3e^{3x}[1+1+3x] = 0 \qquad x = \frac{-2}{3}$ (Inflection point).

Sign diagram of $y'' = 3e^{3x}[2+3x]$



⇒ The sign diagram confirms its Inflection point. (Not stationary).

(e) Graph.



when $x = -\frac{2}{3}$ $\quad y = -\frac{2}{3} \cdot e^{-2} = \frac{-2}{3e^2}$

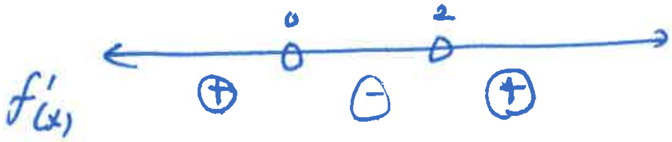
$x = -\frac{1}{3} \quad y = -\frac{1}{3} \cdot e^{-1} = -\frac{1}{3e}.$

$x = 0 \quad y = 0$

#3 (a) $\frac{df}{dx} = 3x^2 - 6x = 0$

$= 3x(x-2) = 0 \qquad x = 0 \qquad x = 2.$

Sign diagram of $f'_{(x)} = 3x(x-2)$ ⇒ decreasing Interval. $\boxed{(0, 2)}$



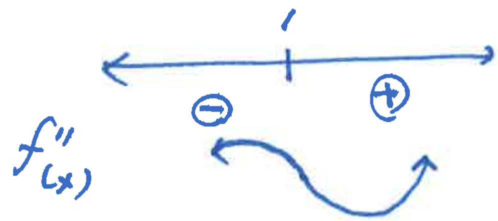
OR $0 < x < 2$

(b) $\frac{d^2f}{dx^2} = 6x - 6 = 0$

$6(x-1) = 0 \qquad x = 1$ ⇒ $\boxed{P(1, 2)}$

$f(1) = 1 - 3 + 4 = 2$

Sign diagram of $f''_{(x)} = 6(x-1)$



#4. When $x = 0$ $\quad y = 1$ ⇒ $\boxed{y = -0.731x + 1}$

Use calculator ⇒ $\frac{dy}{dx}\Big|_{x=0} \approx \underline{1.37}$

Normal slope ≈ -0.731.

#5. $x^2 - 5xy + y^2 = 7$

a) $2x - 5y - 5x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx}[2y - 5x] = 5y - 2x$

$\frac{dy}{dx} = \left[\frac{5y - 2x}{2y - 5x}\right]$

(b) $\frac{dy}{dx}\Big|_{(6,1)} = \frac{5(1) - (2)(6)}{2(1) - 5(6)}$

$= \frac{5 - 12}{2 - 30} = \frac{-7}{-28} = \frac{1}{4}$

Slope of normal line: -4

⇒ $\boxed{y - 1 = -4(x - 6)}$

#6. (a) $\arctan x^2 + \arctan y^2 = \frac{\pi}{4}$

$$\frac{2x}{1+x^4} + \frac{2y}{1+y^4}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\left(\frac{\cancel{2}x}{1+x^4}\right)\left(\frac{1+y^4}{\cancel{2}y}\right) = \boxed{\frac{-x(1+y^4)}{y(1+x^4)}}$$

$x = \frac{1}{\sqrt{2}} \Rightarrow x^2 = \frac{1}{2} \Rightarrow \arctan\left(\frac{1}{2}\right) + \arctan y^2 = \frac{\pi}{4}$

(b)

$$\tan(x^2+y^2) = \frac{\tan x^2 + \tan y^2}{1 - \tan x^2 \tan y^2} = \frac{\frac{1}{2}+y^2}{1-\frac{1}{2}\cdot y^2} = \tan\frac{\pi}{4} = 1$$

$\frac{1}{2} + y^2 = 1 + \frac{1}{2}y^2$

$\frac{3}{2}y^2 = \frac{1}{2}$ $\quad y = \pm\frac{1}{\sqrt{3}}$ $\quad y = \frac{1}{\sqrt{3}}.\ (y>0)$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{\sqrt{2}},\ y=\frac{1}{\sqrt{3}}} \Rightarrow \frac{-\frac{1}{\sqrt{2}}\left(1+\frac{1}{9}\right)}{\frac{1}{\sqrt{3}}\left(1+\frac{1}{4}\right)} = \frac{\sqrt{3}}{\sqrt{2}}\,\frac{\left(\frac{\cancel{10}^{\,2}}{9}\right)}{\left(\frac{\cancel{5}}{4}\right)} = \boxed{\frac{\sqrt{3}\cdot 2}{\sqrt{2}\cdot 9}} = \frac{-2\sqrt{3}}{9\sqrt{2}}$$

#7.

(a) $$\frac{df}{dx} = \frac{1(x^2+1) - 2x(x+1)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2}$$

(b) $-x^2-2x+1=0 \Rightarrow x^2+2x-1=0$ $\quad x = \frac{-2\pm\sqrt{(2)^2+4}}{2}$

$$= \frac{-2\pm\sqrt{8}}{2} = \boxed{-1\pm\sqrt{2}}$$

continue ⇒

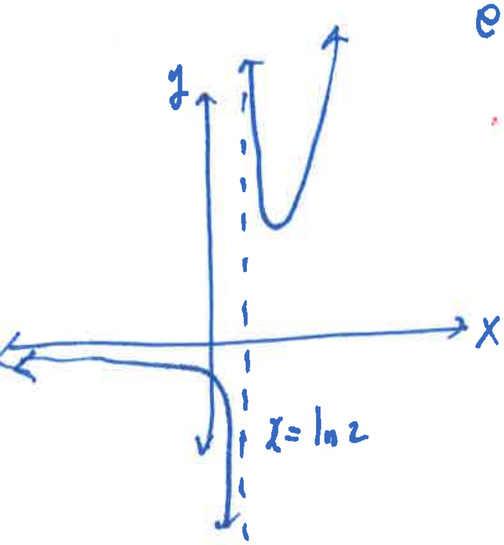
#8 $f(x) = \dfrac{e^{2x}+1}{e^x - 2}$ ⑥

(a) V. Asymptote: $e^x = 2$ $\boxed{x = \ln 2}$

H. Asymptote.

$$\begin{array}{r} e^x + 2 \\ e^x - 2 \,\big)\, (e^x)^2 + 1 \\ \underline{(e^x)^2 - 2e^x} \\ 2e^x + 1 \\ \underline{2e^x - 4} \\ 5 \end{array}$$

$\Rightarrow$ $\boxed{y = 0}$



(b) (i) $\dfrac{df}{dx} = \dfrac{2e^{2x}(e^x - 2) - (e^{2x}+1)(e^x)}{(e^x - 2)^2} = \dfrac{2e^{3x} - 4e^{2x} - e^{3x} - e^x}{(e^x-2)^2}$

$= \boxed{\dfrac{e^{3x} - 4e^{2x} - e^x}{(e^x-2)^2}}$

(ii) $\dfrac{e^{3x} - 4e^{2x} - e^x}{(e^x-2)^2} = 0 \Rightarrow e^{3x} - 4e^{2x} - e^x = 0$

$e^x((e^x)^2 - 4e^x - 1) = 0$

$\Rightarrow e^x = \dfrac{4 \pm \sqrt{16+4}}{2} = \dfrac{4 \pm 2\sqrt{5}}{2}$

$= 2 \pm \sqrt{5}$

$e^x = 2 + \sqrt{5}$ or $\cancel{e^x = 2 - \sqrt{5}}$

No solution.

$\boxed{x = \ln(2+\sqrt{5})}$ is where $f(x)$ has a horizontal tangent line.

(b) (iii) $x = \ln(2+\sqrt{5}) \approx 1.44$
$y \approx 8.47$ $\Rightarrow (1.44, 8.47)$.

(c) $f(0) = \dfrac{e^0+1}{e^0-2} = \dfrac{2}{-1} = -2$.

$\left.\dfrac{df}{dx}\right|_{x=-2} = \dfrac{e^{3(0)} - 4e^{2(0)} - e^{(0)}}{(e^{0}-2)^2} \approx -4$ -4 (tangent gradient)

$\Rightarrow \dfrac{1}{4}$ (Normal gradient)

L_1: $\boxed{y = \frac{1}{4}(x) - 2}$ $\frac{1}{4}$

(d) $\boxed{\dfrac{1}{4} = \dfrac{e^{3x} - 4e^{2x} - e^{x}}{(e^x-2)^2}}$ $\Rightarrow$ Solve by G.C

$x \approx \cancel{2.85}$ $\boxed{1.46}$
$y \approx \cancel{19.6}$. $\boxed{8.47}$

$\left.\dfrac{df}{dx}\right|_{x} \approx \dfrac{1}{4}$.

$L_2 \Rightarrow \boxed{y - 1.96 = 16.9(x - 2.85)}$

9. a) $\dfrac{dy}{dx} = \dfrac{(\cos x - x\sin x)(x+\cos x) - x\cos x(1-\sin x)}{(x+\cos x)^2}$

$= \dfrac{x\cancel{\cos x} \overset{+\cos^2 x}{} - x^2\sin x - x\cancel{\sin x}\cos x - x\cancel{\cos x} + x\cancel{\sin x}\cos x}{(x+\cos x)^2}$

$= \boxed{\dfrac{\cos^2 x - x^2 \sin x}{(x+\cos x)^2}}$

Continue $\Rightarrow$

8

b) $$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \frac{\left(\cos\left(\frac{\pi}{2}\right)\right)^2 - \left(\frac{\pi}{2}\right)^2\left(\sin\frac{\pi}{2}\right)}{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right)^2} = \frac{-\left(\frac{\pi}{2}\right)^2}{\left(\frac{\pi}{2}\right)^2} = -1$$

$$\boxed{y = -\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}}$$