Chapter 21: ZB Questions

#1. a) $\int (1+\tan^2 x)\,dx = \int \sec^2 x\,dx = \tan x + C$

b) $\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

#2. a) $y = \dfrac{3x-2}{2x-1} \Rightarrow 2xy - y = 3x - 2 \Rightarrow 2xy - 3x = y - 2$

$\Rightarrow x\,[2y-3] = y-2$

$x = \dfrac{y-2}{2y-3} \Rightarrow f^{-1}(x) = \dfrac{x-2}{2x-3}$

b) $f(x) = \dfrac{3x-2}{2x-1}$

$= \frac{3}{2} - \frac{1}{2}\left(\frac{1}{2x-1}\right)$

$$\begin{array}{r} \frac{3}{2} \\ 2x-1 \overline{\smash{)}\,3x-2} \\ -\underline{\left(3x-\frac{3}{2}\right)} \\ -\frac{1}{2} \end{array}$$

$\frac{3}{2} - 2 = \frac{-1}{2}$

c) $\Rightarrow \int \dfrac{3x-2}{2x-1}\,dx = \int \left[\frac{3}{2} - \frac{1}{2}\left(\frac{1}{2x-1}\right)\right] dx$

$= \frac{3}{2}x - \frac{1}{2}\cdot\frac{1}{2}\ln|2x-1| + C$

$= \frac{3}{2}x - \frac{1}{4}\ln|2x-1| + C$

#3. $\int \left(\frac{e^x}{e^{2x} + 6e^x + 13} \right) dx$

$$= \int \frac{e^x}{((e^x)^2 + 6e^x + 9) + 4} dx \qquad u = e^x + 3$$

$$du = e^x \cdot dx$$

$$= \int \frac{e^x}{(e^x+3)^2 + 4} dx$$

$$= \int \frac{1}{u^2+4} du \qquad w = \frac{u}{2}$$

$$2\,dw = du$$

$$= \int \frac{1}{4} \left(\frac{1}{(\frac{u}{2})^2 + 1} \right) du$$

$$= \int \frac{1}{\cancel{4}_{2}} \frac{1}{w^2+1} (\cancel{2} \cdot dw) = \frac{1}{2} \arctan\left(\frac{u}{2}\right) + c$$

$$= \boxed{\frac{1}{2} \arctan\left(\frac{e^x+3}{2} \right) + c}$$

#4. $t = \tan x$

$dt = \sec^2 x\, dx$

$$\int \frac{dx \cdot \sec^2 x}{1 + \sin^2 x \cdot \sec^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{\sec^2 x + (\sin^2 x \cdot \frac{1}{\cos^2 x})}$$

$$= \int \frac{\sec^2 x\, dx}{1 + \tan^2 x + \tan^2 x} = \int \frac{dt}{2t^2+1} = \int \frac{dt}{(\sqrt{2}t)^2 + 1}$$

$\rightarrow$

$u = \sqrt{2}t$

$\frac{1}{\sqrt{2}} du = dt$

$$\Rightarrow \int \frac{1}{\sqrt{2}} \frac{du}{u^2+1} = \frac{1}{\sqrt{2}} \arctan[\sqrt{2}t] + C$$

$m = \frac{1}{\sqrt{2}} \qquad n = \sqrt{2}$.

#5 a) $\cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$. $\Rightarrow \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$ where $0 \le \theta \le \frac{\pi}{2}$

If $\theta = \frac{1}{2}x \Rightarrow \cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$. $\quad 0 \le x \le \pi$

b) $\Rightarrow \cos 2\theta = 1 - 2\sin^2\theta$ (Double Angle identy)

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin\theta = \sqrt{\frac{(1-\cos 2\theta)}{2}}$ where $0 \le \theta \le \frac{\pi}{2}$

If $\theta = \frac{1}{2}x \Rightarrow \sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}} \quad 0 \le x \le \pi$

c) $$\int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x} + \sqrt{1-\cos x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sqrt{2}\cos\tfrac{1}{2}x + \sqrt{2}\sin\tfrac{1}{2}x\right) dx$$

$$= 2\sqrt{2}\left[\sin\tfrac{1}{2}x - \cos\tfrac{1}{2}x\right]_{x=0}^{x=\frac{\pi}{2}} = 2\sqrt{2}\left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4} - \sin 0 + \cos 0\right]$$

$= 2\sqrt{2}$

#6. $\displaystyle\int_{-1}^{0} \frac{x+1}{x^2+1}\,dx$

$$= \int_{-1}^{0} \frac{x}{x^2+1}\,dx + \int_{-1}^{0} \frac{1}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x=0 \quad u=1$

$x=-1 \quad u=2.$

$$\int_{2}^{1} \frac{1}{2}\,\frac{1}{u}\,du + \arctan x \Big|_{x=-1}^{x=0}$$

$$= \frac{1}{2}\ln u \Big|_{u=2}^{u=1} + \underbrace{\cancel{\arctan 0}}_{0} + \arctan(1)$$

$$= \underbrace{\cancel{\tfrac{1}{2}\ln 1}}_{0} - \frac{1}{2}\ln 2 + \frac{\pi}{4} = \boxed{\frac{\pi}{4} - \ln\sqrt{2}} \qquad \boxed{\begin{matrix} a=4 \\ b=2 \end{matrix}}$$

#7. $\displaystyle\int_{1}^{2} \left((x-2)^2 + \frac{1}{x}\,\sin \pi x\right)dx$

$$= \frac{1}{3}(x-2)^3 + \ln x - \frac{1}{\pi}\cos \pi x \Bigg|_{x=1}^{x=2}$$

$$= \Big[\frac{1}{2}\underbrace{(2-2)^3}_{0} + \ln 2 - \frac{1}{\pi}\cos 2\pi\Big] - \Big[\frac{1}{3}(1-2)^3 + \underbrace{\cancel{\ln 1}}_{0} - \frac{1}{\pi}\cos\pi\Big]$$

$$= \cancel{-\frac{1}{\pi}} + \frac{1}{3} + \cancel{\frac{1}{\pi}} = \boxed{\frac{1}{3} + \ln 2 - \frac{2}{\pi}}$$

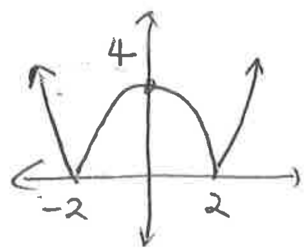
#8. $\displaystyle\int_1^e \frac{(\ln x)^3}{x}\,dx$ $\qquad \ln x = u \qquad x=e \Rightarrow u=1$

$du = \frac{1}{x}\,dx \qquad x=1 \Rightarrow u=0$

$$\Rightarrow \int_0^1 u^3\,du$$

$$= \frac{1}{4}u^4 \Big|_{u=0}^{u=1} = \boxed{\frac{1}{4}}$$

#9. $y = |x^2-4| \Rightarrow \begin{cases} y = x^2-4 & 2 \le x \le 4 \\ y = 4-x^2 & 0 \le x \le 2 \end{cases}$



$$\Rightarrow \int_0^2 (4-x^2)\,dx + \int_2^4 (x^2-4)\,dx$$

$$= \left[4x - \frac{1}{3}x^3\right]\Big|_{x=0}^{x=2} + \left[\frac{1}{3}x^3 - 4x\right]_{x=2}^{x=4}$$

$$\begin{array}{r} 64 \\ -16 \\ \hline 48 \end{array}$$

$$= \left[\cancel{8} - \frac{8}{3}\right] + \left[\frac{64}{3} - \cancel{16}\right] - \left[\frac{8}{3} - \cancel{8}\right] = \frac{64}{3} - \frac{16}{3} = \frac{48}{3} = \boxed{16}$$

#11

$u = x+2 \Rightarrow x = u-2$

$du = dx$

$$\Rightarrow \int \frac{(u-2)^3}{u^2}\,du$$

$$= \int \frac{u^3 - 6u^2 + 12u - 8}{u^2}\,du$$

$$= \int \left(u - 6 + \frac{12}{u} - \frac{8}{u^2}\right)du = \frac{1}{2}(x+2)^2 - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + C$$

\# 12.

a) $$\int \frac{\sin\theta}{1-\cos\theta}\,d\theta = -\int \frac{du}{u}$$

$$= \boxed{\ln|1-\cos\theta| + C}$$

$u = 1-\cos\theta$

$du = \sin\theta\,d\theta$

b) $$\ln[1-\cos a] - \underbrace{\cancel{\ln[1-\cos\tfrac{\pi}{2}]}}_{0}$$

$$= \ln[1-\cos a] = \frac{1}{2}.$$

$1-\cos a = \sqrt{e}$

$\cos a = 1-\sqrt{e}$ $\qquad a = \arccos[1-\sqrt{e}]$

use cal. $\boxed{a \approx 2.28}$

$\frac{\pi}{2} < a < \pi$