$\lim_{n\to\infty} \int_1^n \frac{\ln x}{x}\,dx$ key

IB Math 3:

Name: ______________ Period: _______

Direct Comparison and Limit Comparison Test

Warm up

To prove the series' convergence, determine which test, Divergence test or Integral test, you need to apply for the following series. Do not perform the test.

1. $\sum_{k=1}^{\infty}\left(\frac{k}{2k+5}\right)$: Divergence test. 2. $\sum_{k=1}^{\infty}\left(\frac{\ln k}{k}\right)$: Integral test 3. $\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k}}\right)$: Integral test.

Part I: Direct Comparison Test

> **Suppose** $0 \le a_k \le c_k$ **for** $k = 1, 2, 3, 4 \ldots\ldots$
>
> **1) If** $\sum_{k=1}^{\infty} c_k$ **converges, then** $\sum_{k=1}^{\infty} a_k$ **also converges.**
>
> **Suppose** $0 \le d_n \le a_k$ **for** $k = 1, 2, 3, 4 \ldots\ldots$
>
> **2) If** $\sum_{k=1}^{\infty} d_k$ **diverges, then** $\sum_{k=1}^{\infty} a_k$ **also diverges.**

Notes: Strategy is "looking for a general appearance of a_k and you need to know how to prove the convergence or divergence of $\sum_{k=1}^{\infty} b_k$"

Example 1) Test the series $\sum_{k=1}^{\infty}\frac{1}{3^k+1}$ **for convergence.**

① $\sum \frac{1}{3^k+1} < \sum \frac{1}{3^k}$ ②

Solution:

$\sum \frac{1}{3^k}$ converges by G. Series $r = \frac{1}{3}$

$\boxed{\therefore \sum_{k=1}^{\infty} \frac{1}{3^k+1} \text{ converges}}$

Example 1) Test the series $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}-1}$ **for convergence.**

① $\sum \frac{1}{\sqrt{k}-1} > \sum \frac{1}{\sqrt{k}}$ ②

Solution: Integral test $\frac{1}{\sqrt{k}} = f\ \frac{1}{\sqrt{x}}$

$\Rightarrow \lim_{n\to\infty} \int_1^n x^{-\frac{1}{2}}\,dx = \lim_{n\to\infty} 2\sqrt{x}\Big|_{x=1}^{x=n}$

$= \lim_{n\to\infty} 2\sqrt{n} - 2 = \infty$

diverge

$\boxed{\therefore \sum \frac{1}{\sqrt{k}-1} \text{ diverges}}$

Part II: Limit Comparison Test

Suppose $a_k \geq 0$ **and** $b_k \geq 0$ **for** $k = 1, \quad 2, \quad 3, \quad 4 \ldots\ldots$ **and** $\lim_{k \to \infty} \frac{a_k}{b_k} = L$ where L is finite and positive ($0 < L < \infty$)

1) If $\sum_{k=1}^{\infty} b_k$ **converges, then** $\sum_{k=1}^{\infty} a_k$ **also converges.**

2) If $\sum_{k=1}^{\infty} b_k$ **diverges, then** $\sum_{k=1}^{\infty} a_k$ **also diverges.**

Notes: Strategy is "looking for a general appearance of a_k and you need to know how to prove the convergence or divergence of $\sum_{k=1}^{\infty} b_k$

Example 1) Test the series $\sum_{k=1}^{\infty} \frac{1}{2^k - 5}$ for convergence. $\Rightarrow \sum \frac{1}{2^k - 5} > \sum \frac{1}{2^k}$

Solution: $a_k = \frac{1}{2^k - 5}$ $\qquad b_k = \frac{1}{2^k}$.

Limit comparison : $\lim_{k \to \infty} \frac{\frac{1}{2^k - 5}}{\frac{1}{2^k}} = \lim_{k \to \infty} \frac{2^k \div 2^k}{2^k - 5 \div 2^k}$

$\sum \frac{1}{2^k}$ converges G-series $r = \frac{1}{2}$. $\qquad = \lim_{k \to \infty} \frac{1}{1 - \frac{5}{2^k}\to 0} = 1$

$\therefore \sum \frac{1}{2^k - 5}$ converges

$\therefore \sum \frac{k}{k^2+3k+9}$ diverges

Practice)

1) $\sum_{k=1}^{\infty} \left(\frac{5}{k^4 + 6} \right)$ compare with $\sum \frac{5}{k^4}$

$\sum \frac{5}{k^4+6} < \sum \frac{5}{k^4}$

Int. test: $\lim_{k \to \infty} \int_1^k 5 \cdot x^{-4} dx = \lim_{k \to \infty} \frac{-5}{3} x^{-3}$

$\lim_{k \to \infty} \left[-\frac{5}{3} \frac{1}{k^3} + \frac{5}{3} \right] = \frac{5}{3}$ converges

$\therefore \sum \frac{5}{k^4+6}$ converges.

2) i) $\sum_{k=1}^{\infty} \left(\frac{k}{k^2 + 3k + 9} \right)$ compare with $\sum \frac{k}{k^2+3k}$

① $\sum \frac{k}{k^2+3k+9} < \sum \frac{k}{k^2+3k}$

$\sum \frac{1}{k^2+3k+9} < \sum \frac{1}{k+3}$

② Int. test $\Rightarrow \lim_{k \to \infty} \int_1^k \frac{1}{x+3} dx$

$= \lim_{k \to \infty} \ln(k+3) - \ln 4$

③ $\lim_{k \to \infty} \frac{\frac{k}{k^2+3k+9}}{\frac{k}{k^2+3k}} = \lim_{k \to \infty} \frac{k^3+3k^2}{k^3+3k^2+9k} =$