H.W Answers for Differentiability & continuity

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$
$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h\to 0} \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$
$$\therefore \quad f'(x) = \cos x \times 0 - \sin x \times 1$$
$$= -\sin x$$

$$\left\{\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \quad \{\textbf{Example 6}\}\right\}$$

2 $f(5) = 0$ is defined.

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}(5-x) = 0$ and

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}(x-5) = 0$

$\therefore \lim_{x\to 5} f(x) = 0 = f(5)$

Thus $f(x)$ is continuous at $x = 5$.

Now $f(x) = \begin{cases} x-5, & x \geqslant 5 \\ 5-x, & x < 5\end{cases}$

$\therefore \; f'_-(x) = -1$ and $f'_+(x) = 1$

Hence $f'_-(5) = -1$ and $f'_+(5) = 1$, $\therefore \; f'_-(5) \neq f'_+(5)$

$\Rightarrow f(x)$ is not differentiable at $x = 5$

3 $f(x) = \begin{cases} x+2, & x \geqslant 0 \\ x^2 + 3x, & x < 0\end{cases}$

has an essential 'jump' discontinuity at $x = 0$

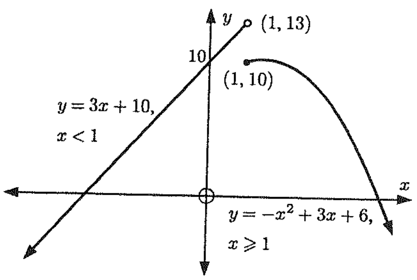
$\therefore \; f(x)$ is not continuous at $x = 0$

$\Rightarrow f(x)$ is not differentiable at $x = 0$

Note: $f'_+(0) = 0$ and $f'_-(0) = 2(0) + 3 = 3$ and $f'_+(0) \neq f'_-(0)$

4 $f(x) = \begin{cases} -x^2 + 5x + 6, & x \geqslant 1 \\ 3x + 10, & x < 1\end{cases}$

a

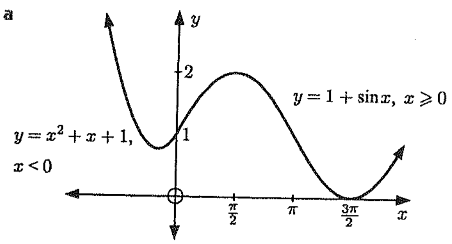


b **i** $f'_-(1) = 3$

ii $f'_+(1) = -2(1) + 5 = 3$

c No, although $f'_-(1) = f'_+(1)$, $f(x)$ is not continuous at $x = 1$.

5 **a**



$f(0) = 1 + \sin 0 = 1$ is defined

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2 + x + 1) = 1$

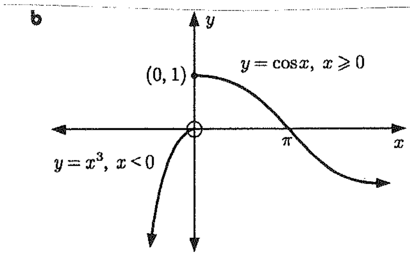
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-}(1 + \sin x) = 1$

$\therefore \lim_{x\to 0} f(x) = 1 = f(0)$

$\Rightarrow f(x)$ is continuous at $x = 0$

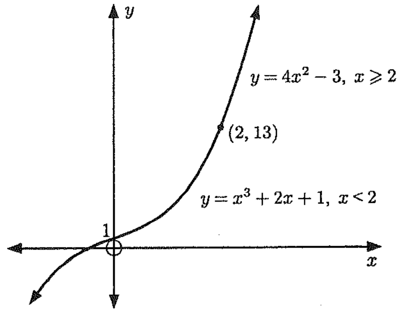
Now $f'_-(0) = 2(0) + 1 = 1$ and $f'_+(0) = \cos 0 = 1$

b



As $f(x)$ is discontinuous at $x = 0$, it is not differentiable at $x = 0$.

c



$f(2) = 4(2)^2 - 3 = 13$ is defined

$\lim_{x\to 2^-} f(x) = 2^3 + 2(2) + 1 = 13$

$\lim_{x\to 2^+} f(x) = 4(2)^2 - 3 = 13$

$\therefore \lim_{x\to 2} f(x) = 13 = f(2)$

$\Rightarrow f(x)$ is continuous at $x = 2$

Now $f'_-(2) = 3(2)^2 + 2 = 14$ and

$f'_+(2) = 8(2) = 16$

$\therefore \; f'_-(2) \neq f'_+(2)$

$\Rightarrow f(x)$ is not differentiable at $x = 2$

6 $f(0) = k \sin 0 = 0$

$\therefore \; f(0)$ is defined.

Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(\tan x) = \tan 0 = 0$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(k \sin x) = k(0) = 0$

$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$

$\Rightarrow f(x)$ is continuous at $x = 0$ for all $k \in \mathbb{R}$.

Now $f'_-(0) = \sec^2(0) = \frac{1}{1^2} = 1$ and

$f'_+(0) = k\cos(0) = k$

$\therefore \; f'_-(0) = f'_+(0) \;\Leftrightarrow\; k = 1$

$\therefore \; f(x)$ is differentiable at $x = 0 \;\Leftrightarrow\; k = 1$

7 **a** $f(x) = \begin{cases} x^2, & x \leqslant 1 \\ cx + d, & x > 1\end{cases}$

$f(1) = 1^2 = 1$ is defined.

Also, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(cx + d) = c + d$

$\therefore \lim_{x\to 1} f(x) = f(1) \;\Leftrightarrow\; c + d = 1$ .... (1)

Now $f'_-(1) = 2(1) = 2$ and $f'_+(1) = c$

$\therefore \; f'(1)$ exists $\Leftrightarrow c = 2$

But from (1), $c + d = 1$ $\therefore \; d = -1$

$\therefore \; c = 2, \; d = -1$

b $f(x) = \begin{cases} \sin(x-1) + cx, & x \geqslant 1 \\ x^2 - x + d, & x < 1\end{cases}$

$f(1) = \sin 0 + c = c$ is defined.

Also, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2 - x + d) = d$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(\sin(x-1) + cx) = c$

$\therefore \lim_{x\to 1} f(x) = f(1) \;\Leftrightarrow\; c = d$ .... (1)

Now $f'_-(1) = 2(1) - 1 = 1$

and $f'_+(1) = \cos(0) + c(1) = 1 + c$

$\therefore \; f'(1)$ exists $\Leftrightarrow 1 + c = 1 \;\Leftrightarrow\; c = 0$

Thus, from (1), $c = d = 0$