Key for Differential Equations (IB Exam Style Questions) ①

#1. $\frac{dy}{\cos^2 y} = \cos x \, dx$

$\Rightarrow \int \sec^2 y \cdot dy = \int \cos x \, dx$

$\tan y = \sin x + c \Leftarrow x = \pi \quad y = \frac{\pi}{4}$

$\tan \frac{\pi}{4} = \sin \pi + c \Rightarrow \boxed{c = +1}$

$\Rightarrow y = \tan^{-1}(\sin x + 1)$ OR $\arctan(\sin x + 1)$

#2. $\frac{d}{dx} e^x(\sin x + \cos x)$

$= e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x.$

$\int \cancel{d} e^x(\sin x + \cos x) = \int 2e^x \cos x \, dx.$

$\frac{1}{2} 2 \boxed{\int e^x \cos x \, dx} = \int \frac{1}{2} \cancel{2} e^x (\sin x + \cos x) + c$

$\Rightarrow \int e^x \cos x \, dx = e^x(\sin x + \cos x) + c$

#3. $\frac{dy}{dx} = \frac{y^2}{1+x} \quad x > -1$

a) $\int \frac{dy}{y^2} = \int \frac{1}{(1+x)} dx \Rightarrow \frac{-1}{y} = \ln(1+x) + c \in (0, 1)$

$-1 = c$

$\frac{-1}{y} = \ln(1+x) - 1 \Rightarrow y = \frac{1}{1 - \ln(1+x)}.$

b) $y \to \infty$ as $x \to a$ $\Rightarrow$ Vertical Asymptote.

$$1 - \ln(1+x) = 0 \qquad \ln(1+x) = 1$$

$$1 + x = e$$

$$\boxed{x = e - 1}$$

#4.

$$\frac{dy}{dx} + y \cdot \tan x = \cos^2 x \Leftarrow \text{linear differential Equation}$$

$$I = e^{\int \tan x \, dx} = e^{-\ln(\cos x)} = \sec x.$$

$$\underbrace{\sec x \cdot y' + y \cdot \tan x \cdot \sec x}_{} = \cos^2 x \cdot \sec x = \cos x$$

$$d(\sec x \cdot y) = \int \cos x \, dx$$

$$\sec x \cdot y = \sin x + c$$

$$y = \frac{\sin x}{\sec x} + \frac{c}{\sec x} = \sin x \cos x + c \cdot \cos x$$

$$x = 0 \quad y = 2$$

$$2 = \sin 0 \cancel{\cos}(x) + c \cos x \Rightarrow c = 2$$

$$y = \sin x \cos x + 2 \cos x \quad \text{OR} \quad \cos x(\sin x + 2)$$

#5. $t\frac{dy}{dt} = \cos t - 2y$

$I = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

$t\frac{dy}{dt} + 2y = \cos t$

$\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos t}{t}$ (linear D.Q)

$\underbrace{t^2 \cdot y' + 2t \cdot y}_{} = t\cos t$

$\frac{d}{dx}(t^2 \cdot y) = t\cos t$

$\int d(t^2 \cdot y) = \int t\cos t\, dt$

$u = t \quad dv = \cos t$

$\oplus$

$du = 1dt \quad v = \sin t$

$\ominus$

$\Rightarrow t^2 y = t\sin x - \int \sin t\, dt$

$y = \frac{1}{t^2}\left[t\sin x + \cos t + c\right]$

$y = \frac{\sin x}{t} + \frac{\cos t}{t^2} + \frac{c}{t^2}$

#6. $\frac{dy}{dx} = 2e^x + y\tan x$

$\frac{dy}{dx} - \tan x \cdot y = 2e^x$ (Linear D.Q)

$I = e^{\int -\tan x\, dx} = e^{\ln\cos x} = \cos x$.

$$\cos x \cdot y' - \tan x \cdot \cos x \cdot y = 2\cos x \cdot e^x$$

$$\underbrace{\cos x \, y' - \sin x}\, y = 2\cos x \cdot e^x$$

$$\int \frac{d}{dx}\left(y \cdot \cos x\right) = \int 2\cos x \cdot e^x \, dx$$

u	dv
$2e^x$	$\cos x$
$2e^x$ (+)	$\sin x$
$2e^x$ (−)	$-\cos x$
(+)	

$$y \cos x = e^x \sin x$$

$$\int 2\cos x \cdot e^x dx = 2e^x \sin x + 2e^x \cos x - \int 2e^x \cos x \, dx$$

$$2\int 2\cos x \, e^x dx = \frac{\cancel{2}e^x \sin x}{\cancel{2}} + \frac{\cancel{2}e^x \cos x}{\cancel{2}}$$

$$\int 2\cos x \cdot e^x dx = e^x \sin x + e^x \cos x + C.$$

$$\Rightarrow \quad y\cos x = e^x \sin x + e^x \cos x + C \quad \Leftarrow (0, 1)$$

$$1 = 1 + C \qquad C = 0.$$

$$\Rightarrow \boxed{y = e^x \frac{\sin x}{\cos x} + e^x} \quad \text{OR} \quad \boxed{y = e^x \tan x + e^x}$$