18F: Derivatives of Exponential Functions Investigation

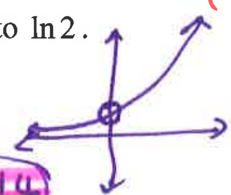
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(h)}{h}.$$

1. Given $f(x)=2^x$, use first principle to show that $f'(x)=2^x\left(\lim_{h\to 0}\frac{2^h-1}{h}\right)$.

$$f'(x) = \lim_{h\to 0}\frac{2^{x+h}-2^x}{h} = \lim_{h\to 0}\frac{2^x 2^h - 2^x}{h} = \lim_{h\to 0}\frac{2^x(2^h-1)}{h} = \boxed{2^x\left(\lim_{h\to 0}\frac{2^h-1}{h}\right)}$$

2. Evaluate $\lim_{h\to 0}\frac{2^h-1}{h}$ accurate to four decimal places using your GFC. Compare this value to $\ln 2$.

$\left(\frac{0}{0}\right)$



h	$-.00001$	0	0.00001
$\frac{2^h-1}{h}$	$.69314$	undef	$.69314$

$$\Rightarrow \lim_{h\to 0}\frac{2^h-1}{h} \approx .69314.$$

$$\ln 2 \approx .693147\ldots$$

3. Hence if $f(x)=2^x$, then what can you say about $\frac{df}{dx}$.

$$\frac{d}{dx}2^x = \lim_{h\to 0}\frac{2^{x+h}-2^h}{h} = 2^x\lim_{h\to 0}\frac{2^h-1}{h} = \boxed{2^x \cdot \ln 2}$$

4. Given $f(x)=e^x$, use first principle to show that $f'(x)=e^x\left(\lim_{h\to 0}\frac{e^h-1}{h}\right)$.

$$f'(x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = e^x\left(\lim_{h\to 0}\frac{e^h-1}{h}\right)$$

5. Evaluate $\lim_{h\to 0}\frac{e^h-1}{h}$ accurate to four decimal places using your GFC. Compare this value to $\ln e = 1$.

h	$-.00001$	0	$.00001$
$\frac{e^h-1}{h}$	1	undef.	1

$$\Rightarrow \lim_{h\to 0}\frac{e^h-1}{h} = 1$$

6. Hence if $f(x)=e^x$, then what can you say about $\frac{df}{dx}$.

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = \boxed{e^x}$$

Summary: $\frac{d}{dx}(b^x) = b^x \cdot \ln b$. $\frac{d}{dx}(e^x) = e^x \cdot \ln e = \boxed{e^x}$ $\frac{d}{dx}\left(e^{f(x)}\right) = \frac{d e^f}{df} \cdot \frac{df}{dx} = e^f \cdot \frac{df}{dx}$

Examples) Find $\frac{dy}{dx}$ for…

1. $y = 2 \cdot 5^x$

$\frac{dy}{dx} = (2) \cdot 5^x \cdot (\ln 5)$

$= (2)(\ln 5)(5^x)$

2. $y = e^{x^2-3}$

$\frac{dy}{dx} = (e^{x^2-3})(2x)$

3. $y = x^2 e^{5x+7}$

$\frac{dy}{dx} = (x^2)' \cdot e^{5x+7} + x^2 \cdot (e^{5x+7})'$

$= \boxed{2x \cdot e^{5x+7} + 5(x^2)(e^{5x+7})}$

OR $= \boxed{x \cdot e^{5x+7}(2+5x)}$

4. $y = 7^{4x-x^5} = 7^{f(x)}$

$\frac{dy}{dx} = \boxed{(7^{4x-x^5}) \cdot (\ln 7)(4-5x^4)}$

5. $y = \sqrt{x^5 + 5^x} = (x^5+5^x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \boxed{\frac{1}{2}(x^5+5^x)^{-\frac{1}{2}}(5x^4 + 5^x \cdot \ln 5)}$

$= \boxed{\frac{5x^4 + \ln 5 \cdot 5^x}{2\sqrt{x^5+5^x}}}$

6. $y = \frac{1}{\sqrt{2e^{-x}+1}} = (2e^{-x}+1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}}(-2e^{-x})$

$= \boxed{\frac{-1}{e^x(2e^{-x}+1)^{3/2}}}$

7. Find the equation of tangent to the curve $y = 2 + e^{2x}$ at $x = 0$. Find the value of y

$y - y_1 = m(x - x_1)$

1) Find $\frac{dy}{dx}$

$\frac{dy}{dx} = 2e^{2x}$

$\left.\frac{dy}{dx}\right|_{x=0} = 2 \cdot e^0 = 2$

2) $y = 2 + e^0 = 3$

3) $(0, 3)$ $m = 2$

$\boxed{y - 3 = 2(x - 0)}$

$\boxed{y = 2x + 3}$