IB Math HL1 Name: key Period: ____

Differentiability and Continuity

- **Differentiability: Function f is differentiable if and only if it is differentiable at every value of x in its domain.**

A function $f: D \to R$ is differentiable at x=a, $a \in D$, if

1. f is continuous at x=a
2. Suppose f is a real function with domain D containing an open interval about x=a.

f is differentiable at x=a if $f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ (right-hand derivative) and

$f'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$ (left-hand derivative) both exits and are equal.

- **Continuity: If function f is differentiable at x=c, then f is continuous at x=c.**

Example1)

Find the value(s) of the constants a and b what make piecewise function f(x) differentiable at x=2.

$$f(x) = \begin{cases} ax^3 & x \le 2 \\ b(x-3)^2 + 10 & x > 2 \end{cases} \quad x=2$$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) \Rightarrow \lim_{x\to 2^+} b(x-3)^2+10 = \lim_{x\to 2^-} ax^3$

$\boxed{b+10=8a}$ ①

$\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^-} f'(x) \Rightarrow \lim_{x\to 2^+} 2b(x-3) = \lim_{x\to 2^-} 3ax^2$

$\boxed{-2b=12a}$ ②

$b=-6a$

$-6a+10=8a$

$14a=10$

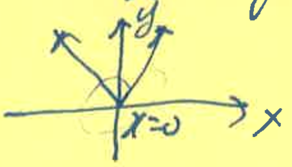
$a=\frac{5}{7}$

$b=\frac{-30}{7}$

Example 2)

Prove that $f(x) = |x|$ is continuous but not differentiable at x=0.

$$f(x) = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$



$\lim_{x\to 0^+} x = \lim_{x\to 0^-} -x$

$0=0$

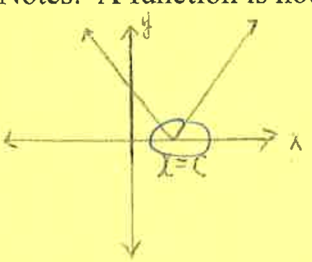
f(x) is continuous at $x=0$

$\lim_{x\to 0^+} 1 \ne \lim_{x\to 0^-} -1$

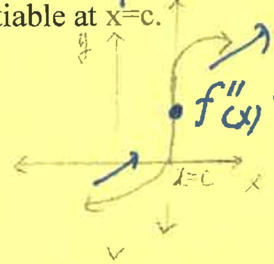
$1 \ne -1$

f(x) is not differentiable at $x=0$.

Notes: A function is not differentiable at x=c.

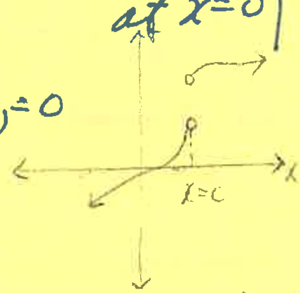


Cusp

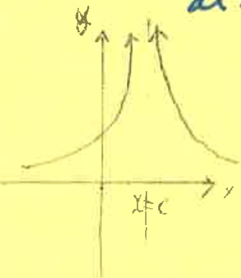


$f''(x)=f'(x)=0$

vertical tangent line



$\lim_{x\to c^+} f(x) \ne \lim_{x\to c^-} f(x)$



f(c) is undefined

Practice) Prove that $f(x)=\begin{cases}\sin x & x\geq 0\\ x^2+5x & x<0\end{cases}$ is continuous but not differentiable at x=0.

$\lim_{x\to 0^+} \sin x = \lim_{x\to 0^-} x^2+5x$

$0 = 0$

$\therefore$ f(x) is continuous.

$\lim_{x\to 0^+} (\sin x)' = \lim_{x\to 0^-} (x^2+5x)'$

$\Rightarrow \lim_{x\to 0^+} \cos x = \lim_{x\to 0^-} 2x+5$

$1 \neq 5$

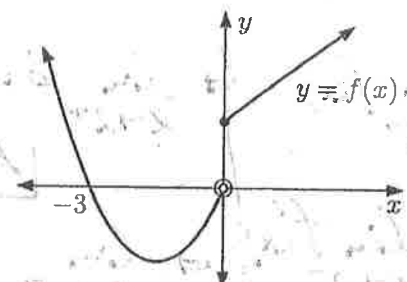
Homework: (Answers are on seperate sheet) $\therefore$ f(x) is Not differentiable.

1 Let $f(x)=\cos x$. Use the definition of the derivative to prove that $f'(x)=-\sin x$.

2 Prove that $f(x)=|x-5|=\begin{cases}x-5, & x\geqslant 5\\ 5-x, & x<5\end{cases}$ is continuous but not differentiable at $x=5$.

3 Let $f(x)=\begin{cases}x+2, & x\geqslant 0\\ x^2+3x, & x<0.\end{cases}$

Explain why $f(x)$ is not differentiable at $x=0$.



4 Let $f(x)=\begin{cases}-x^2+5x+6, & x\geqslant 1\\ 3x+10, & x<1.\end{cases}$

a Sketch the function $y=f(x)$.

b Calculate: **i** $f'_-(1)$ **ii** $f'_+(1)$

c Is f differentiable at $x=1$? Explain your answer.

5 For each of the following functions and the given value of a, determine whether the function is differentiable at $x=a$.

a $f(x)=\begin{cases}1+\sin x, & x\geqslant 0\\ x^2+x+1, & x<0\end{cases}$, $a=0$

b $f(x)=\begin{cases}\cos x, & x\geqslant 0\\ x^3, & x<0\end{cases}$, $a=0$

c $f(x)=\begin{cases}4x^2-3, & x\geqslant 2\\ x^3+2x+1, & x<2\end{cases}$, $a=2$

6 Investigate the continuity and differentiability of f at $x=0$ if

$f(x)=\begin{cases}k\sin x, & x\geqslant 0\\ \tan x, & x<0\end{cases}$, where $k\in\mathbb{R}$ is any constant.

7 Find constants $c, d\in\mathbb{R}$ so that the given function is differentiable at $x=1$.

a $f(x)=\begin{cases}x^2, & x\leqslant 1\\ cx+d, & x>1\end{cases}$

b $f(x)=\begin{cases}\sin(x-1)+cx, & x\geqslant 1\\ x^2-x+d, & x<1\end{cases}$