Review – Differential Calculus 2 (HL)

♦ **Paper 2 Review – 9 Questions** ♦ calculator allowed [**worked solutions** included]

syllabus content: chain rule, product rule, quotient rule, 2nd derivative test for maxima and minima, tangent lines, inflexion points, optimization, implicit differentiation

1. The point $P\left(1, p\right)$, where $p > 0$, lies on the curve $2x^2y + 3y^2 = 16$.

(a) Calculate the value of *p*.

(b) Calculate the gradient of the tangent to the curve at point *P*.

2. Given that $y = \dfrac{x}{e^x} + \sqrt{2x}$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

3. The normal to the curve $y = \dfrac{k}{x} + \ln\left(x^2\right)$, for $x \neq 0$, $k \in \mathbb{R}$, at the point where $x = 2$, has the equation $3x + 2y = c$ where *c* is an unknown constant. Find the exact value of *k*.

4. Find the **exact** coordinates of any maximum or minimum points on the graph of $y = \frac{1}{2}\sin 2x + \cos x$ in the interval $0 \leq x \leq 2\pi$. Clearly identify as a maximum or minimum.

5. Calculate the minimum distance from the point $A\left(2, -\dfrac{1}{2}\right)$ to the parabola $y = x^2$.

6. The function *f* is defined by $f(x) = \dfrac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that $f'(x) = \dfrac{2x - x^2\ln 2}{2^x}$

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(b) (i) Find the **exact** value of *x* satisfying the equation $f'(x) = 0$

(ii) Show that this value gives a maximum value for $f(x)$.

(c) Find the *x*-coordinates of the two points of inflexion on the graph of *f*.

7. Let $f\left(x\right) = \dfrac{x^2 + 5x + 5}{x + 2}$, $x \neq -2$

(a) Find $f'\left(x\right)$

(b) Solve $f'\left(x\right) > 2$

8. Consider the curve with the equation $x^2 + xy + y^2 = 19$.

(a) Find the equation of the line tangent to the curve at the point where $x = -2$ and $y > 0$.

(b) Find the *x*-coordinate of any point(s) where the tangent to the curve is parallel to the *y*-axis.

9. A rectangle is drawn so that its lower vertices are on the *x*-axis and its upper vertices are on the curve $y = e^{-x^2}$. The area of this rectangle is denoted by *A*.

(a) Write down an expression for *A* in terms of *x*.

(b) Find the maximum value of *A*.

Bonus: Find the exact value of the constant *c* in question 3.

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Worked Solutions

1. (a) $2(1)^2y+3y^2=16 \Rightarrow 3y^2+2y-16=0 \Rightarrow (3y+8)(y-2)=0 \Rightarrow y=-\frac{8}{3}$ or $y=2$

Since $p>0$, then $p=2$

(b) $2\frac{d}{dx}(x^2y)+3\frac{d}{dx}(y^2)=\frac{d}{dx}(16) \Rightarrow 2\left(2x\cdot y+x^2\cdot\frac{dy}{dx}\right)+3\left(2y\cdot\frac{dy}{dx}\right)=0$

$\frac{dy}{dx}(2x^2+6y)=-4xy \Rightarrow \frac{dy}{dx}=\frac{-4xy}{2x^2+6y}$

At $(1,2)$: $\frac{dy}{dx}=\frac{-4(1)(2)}{2(1)^2+6(2)}=\frac{-8}{14}=-\frac{4}{7}$

2. $y=\frac{x}{e^x}+\sqrt{2x}=xe^{-x}+(2x)^{1/2}$

$\frac{dy}{dx}=1\cdot e^{-x}+x\cdot e^{-x}(-1)+\frac{1}{2}(2x)^{-1/2}(2)=e^{-x}-x\cdot e^{-x}+(2x)^{-1/2}$

$\frac{dy}{dx}=\frac{1-x}{e^x}+\frac{1}{\sqrt{2x}}$

$\frac{dy}{dx}=(1-x)e^{-x}+(2x)^{-1/2}$

$\frac{d^2y}{dx^2}=-1\cdot e^{-x}+(1-x)e^{-x}(-1)+\left(-\frac{1}{2}\right)(2x)^{-3/2}(2)$

$\frac{d^2y}{dx^2}=-e^{-x}+(x-1)e^{-x}-(2x)^{-3/2}$

$\frac{d^2y}{dx^2}=-e^{-x}+xe^{-x}-e^{-x}-\frac{1}{\sqrt{(2x)^3}}=(x-2)e^{-x}-\frac{1}{\sqrt{8x^3}}$

$\frac{d^2y}{dx^2}=xe^{-x}-\frac{1}{2\sqrt{2x^3}}$

3. $y=kx^{-1}+\ln(x^2) \Rightarrow \frac{dy}{dx}=-kx^{-2}+\frac{1}{x^2}\cdot 2x=-\frac{k}{x^2}+\frac{2}{x}$; when $x=2$: $\frac{dy}{dx}=-\frac{k}{2^2}+\frac{2}{2}=\frac{-k+4}{4}$

$3x+2y=c \Rightarrow y=-\frac{3}{2}x+\frac{c}{2}$; slope of <u>normal</u> is $\frac{2}{3}$, thus $\frac{-k+4}{4}=\frac{2}{3} \Rightarrow -k+4=\frac{8}{3} \Rightarrow k=\frac{4}{3}$

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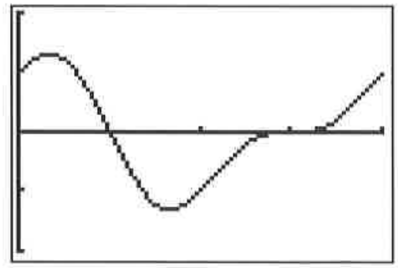
Worked Solutions (continued)

4. $\frac{dy}{dx}=\frac{1}{2}(\cos 2x)\cdot 2-\sin x=\cos 2x-\sin x$; $\frac{d^2y}{dx^2}=-2\sin 2x-\cos x$

substituting $1-2\sin^2 x$ for $\cos 2x$ gives:

$1-2\sin^2 x-\sin x=0 \Rightarrow 2\sin^2 x+\sin x-1=0$

$(2\sin x-1)(\sin x+1)=0 \Rightarrow \sin x=\frac{1}{2}$ or $\sin x=-1$



graph of $y=\frac{1}{2}\sin 2x+\cos x,\ 0\le x\le 2\pi$

- $\sin x=\frac{1}{2} \Rightarrow x=\frac{\pi}{6}$ or $x=\frac{5\pi}{6}$

$$y\left(\frac{\pi}{6}\right)=\frac{1}{2}\sin\left(2\cdot\frac{\pi}{6}\right)+\cos\frac{\pi}{6}=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{4}$$

$$y\left(\frac{5\pi}{6}\right)=\frac{1}{2}\sin\left(2\cdot\frac{5\pi}{6}\right)+\cos\frac{5\pi}{6}=\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}=-\frac{3\sqrt{3}}{4}$$

at $x=\frac{\pi}{6}$: $\frac{d^2y}{dx^2}=-2\sin\left(2\cdot\frac{\pi}{6}\right)-\cos\frac{\pi}{6}=-2\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=-\frac{3\sqrt{3}}{2}<0$; thus, there is a max at $x=\frac{\pi}{6}$

at $x=\frac{5\pi}{6}$: $\frac{d^2y}{dx^2}=-2\sin\left(2\cdot\frac{5\pi}{6}\right)-\cos\frac{5\pi}{6}=-2\cdot\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{\sqrt{3}}{2}\right)=\frac{3\sqrt{3}}{2}>0$; thus, a min at $x=\frac{5\pi}{6}$

- $\sin x=-1 \Rightarrow x=\frac{3\pi}{2}$

at $x=\frac{3\pi}{2}$: $\frac{d^2y}{dx^2}=-2\sin\left(2\cdot\frac{3\pi}{2}\right)-\cos\frac{3\pi}{2}=-2\cdot 0-0=0$; 2^{nd} derivative test is <u>not</u> conclusive

test sign of 1^{st} derivative just before and after $x=\frac{3\pi}{2}\approx 4.71$; $\frac{dy}{dx}=\cos 2x-\sin x$

at $x=4.5$: $\frac{dy}{dx}=\cos(9)-\sin(4.5)\approx 0.0664>0$; at $x=5$: $\frac{dy}{dx}=\cos(10)-\sin(5)\approx 0.120>0$

therefore, no max or min at $x=\frac{3\pi}{2}$

Therefore, maximum at $\left(\frac{\pi}{6},\frac{3\sqrt{3}}{4}\right)$, and minimum at $\left(\frac{5\pi}{6},-\frac{3\sqrt{3}}{4}\right)$

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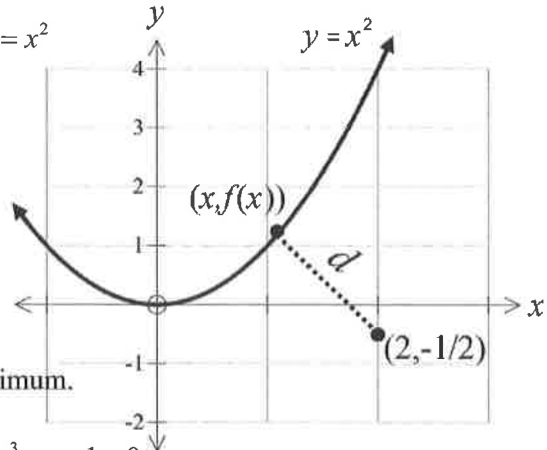
Worked Solutions (continued)

5. let d be the minimum distance from $\left(2,-\frac{1}{2}\right)$ to $f(x)=x^2$



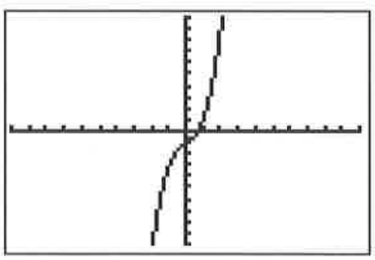
$$d=\sqrt{(x-2)^2+\left(x^2+\tfrac{1}{2}\right)^2}=\sqrt{x^2-4x+4+x^4+x^2+\tfrac{1}{4}}$$

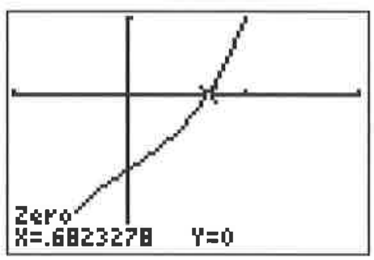
$$=\sqrt{x^4+2x^2-4x+\tfrac{17}{4}}$$

The min value for the function $d=\sqrt{x^4+2x^2-4x+\frac{17}{4}}$ will occur where the function $x^4+2x^2-4x+\frac{17}{4}$ has a minimum.

$$y=x^4+2x^2-4x+\tfrac{17}{4} \Rightarrow \frac{dy}{dx}=4x^3+4x-4=0 \Rightarrow x^3+x-1=0$$

Solve $x^3+x-1=0$ by finding any x-intercept(s) of $y=x^3+x-1$





solution: $x\approx 0.6823278038\ldots$

$$d=\sqrt{0.682^4+2(0.682)^2-4(0.682)+\tfrac{17}{4}}\approx 1.6335813\ldots$$

Thus, the minimum distance from the point $\left(2,-\frac{1}{2}\right)$ to the graph of $y=x^2$ is approximately 1.63

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Worked Solutions (continued)

6. (a) (i) $f(x)=\dfrac{x^2}{2^x} \Rightarrow f'(x)=\dfrac{2^x\cdot 2x-x^2\cdot 2^x\cdot \ln 2}{\left(2^x\right)^2}=\dfrac{\cancel{2^x}\left(2x-x^2\ln 2\right)}{\cancel{2^x}\cdot 2^x}=\dfrac{2x-x^2\ln 2}{2^x}$ ***Q.E.D.***

(ii) $f''(x)=\dfrac{2^x\left(2-2x\ln 2\right)-\left(2x-x^2\ln 2\right)2^x\ln 2}{\left(2^x\right)^2}=\dfrac{\cancel{2^x}\left(2-2x\ln 2-2x\ln 2+x^2\left(\ln 2\right)^2\right)}{\cancel{2^x}\cdot 2^x}$

$$=\frac{2-4x\ln 2+x^2\left(\ln 2\right)^2}{2^x}=\frac{\left(\ln 2\right)^2x^2-\left(4\ln 2\right)x+2}{2^x}$$

(b) (i) $f'(x)=\dfrac{2x-x^2\ln 2}{2^x}=0 \Rightarrow 2x-x^2\ln 2=0 \Rightarrow x\left(2-x\ln 2\right)=0 \Rightarrow$ ~~$x=0$~~ since $x>0$

OR $2-x\ln 2=0 \Rightarrow x=\dfrac{2}{\ln 2}$

(ii) $f''\left(\dfrac{2}{\ln 2}\right)=\dfrac{\left(\ln 2\right)^2\left(\dfrac{2}{\ln 2}\right)^2-\left(4\ln 2\right)\left(\dfrac{2}{\ln 2}\right)+2}{2^{2/\ln 2}}=\dfrac{4-8+2}{e^2}=-\dfrac{2}{e^2}$

<u>note</u>: $2^{2/\ln 2}=\left(2^{1/\ln 2}\right)^2$; let $a=2^{1/\ln 2} \Rightarrow a^{\ln 2}=2 \Rightarrow \log_a 2=\ln 2 \Rightarrow a=e \Rightarrow \left(2^{1/\ln 2}\right)^2=e^2$

since $f''\left(\dfrac{2}{\ln 2}\right)=-\dfrac{2}{e^2}<0$ graph of f is concave down at $x=\dfrac{2}{\ln 2} \Rightarrow$ max value for f at $x=\dfrac{2}{\ln 2}$

(c) solving $f''(x)=0$

polyRoots$\left(\left(\ln(2)\right)^2\cdot x^2-4\cdot\ln(2)\cdot x+2,x\right)$
$\{0.845111188584, 4.92566897497\}$

finding sign (pos. or neg.) of three 'test' points: $x=0.8$, $x=3$ and $x=5$

since sign of $f''(x)$ changes at $x\approx 0.845$ and at $x\approx 4.93$, the graph of f has points of inflexion at $x\approx 0.845$ and $x\approx 4.93$

$f(x):=\dfrac{\left(\ln(2)\right)^2\cdot x^2-4\cdot\ln(2)\cdot x+2}{2^x}$	*Done*
$f(0.8)$	0.051358
$f(3)$	-0.249211
$f(5)$	0.004637

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Worked Solutions (continued)

7. (a) $f'(x)=\dfrac{(x+2)(2x+5)-(1)(x^2+5x+5)}{(x+2)^2}=\dfrac{2x^2+9x+10-x^2-5x-5}{(x+2)^2} \Rightarrow f'(x)=\dfrac{x^2+4x+5}{(x+2)^2}$

(b) $\dfrac{x^2+4x+5}{(x+2)^2}>2 \Rightarrow \dfrac{x^2+4x+5}{(x+2)^2}-2>0 \Rightarrow \dfrac{x^2+4x+5}{(x+2)^2}-\dfrac{2(x+2)^2}{(x+2)^2}>0 \Rightarrow$

$\dfrac{x^2+4x+5-2x^2-8x-8}{(x+2)^2}>0 \Rightarrow \dfrac{-x^2-4x-3}{(x+2)^2}>0 \Rightarrow \dfrac{x^2+4x+3}{(x+2)^2}<0$ Switch inequality sign after multiplying through by −1

denominator $(x+2)^2$ is always positive; thus, $\dfrac{x^2+4x+3}{(x+2)^2}<0$ when $x^2+4x+3<0$

$(x+3)(x+1)<0 \Rightarrow$ (x+3)(x+1): pos. | -3 | neg. | -1 | pos. and $f'(x)=\dfrac{x^2+4x+5}{(x+2)^2}$ undefined at $x=-2$

Therefore, $f'(x)>2$ when $-3<x<-1,\ x\neq -2$

8. (a) $x=-2$: $(-2)^2-2y+y^2=19 \Rightarrow y^2-2y-15=0 \Rightarrow (y-5)(y+3)=0 \Rightarrow y=5$ since $y>0$

Thus, point of tangency is $(-2,5)$; find $\dfrac{dy}{dx}$: $2x+y+x\cdot\dfrac{dy}{dx}+2y\cdot\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{-2x-y}{x+2y}$

at $(-2,5)$: $\dfrac{dy}{dx}=\dfrac{-2(-2)-5}{-2+2\cdot 5}=-\dfrac{1}{8}$; eqn of tangent line: $y-5=-\dfrac{1}{8}(x-(-2)) \Rightarrow y=-\dfrac{1}{8}x+\dfrac{19}{4}$

(b) "parallel to the y-axis" $\Rightarrow$ vertical line; slope is undefined $\Rightarrow$ find where $\dfrac{dy}{dx}$ is undefined

$\dfrac{dy}{dx}=\dfrac{-2x-y}{x+2y}$ is undefined when $x+2y=0 \Rightarrow y=-\dfrac{1}{2}x$; substitute into original equation

$x^2+x\left(-\dfrac{1}{2}x\right)+\left(-\dfrac{1}{2}x\right)^2=19 \Rightarrow x^2-\dfrac{1}{2}x^2+\dfrac{1}{4}x^2=19 \Rightarrow \dfrac{3}{4}x^2=19 \Rightarrow x^2=\dfrac{76}{3}$

Therefore, tangent to the curve is vertical when $x=\pm\sqrt{\dfrac{76}{3}}$ $\left[\text{or } x=\pm 2\sqrt{\dfrac{19}{3}},\ \text{or } x\approx\pm 5.03\right]$

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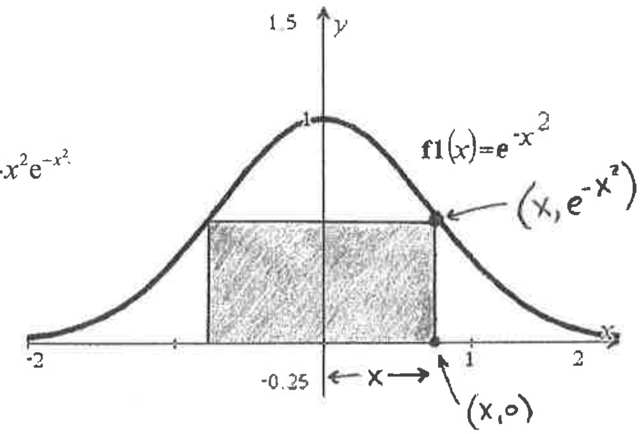
Worked Solutions (continued)



9. (a) $A(x)=2x\cdot e^{-x^2}$

(b) $A'(x)=2e^{-x^2}+2x\left(e^{-x^2}(-2x)\right)=2e^{-x^2}-4x^2e^{-x^2}$

$$=\left(2-4x^2\right)e^{-x^2}=0$$

$e^{-x^2}\neq 0 \quad 2-4x^2=0 \;\Rightarrow\; x=\pm\sqrt{\frac{1}{2}}$

but $x>0$, so $x=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$; maximum area occurs when $x=\frac{\sqrt{2}}{2}$

maximum area is $A\left(\frac{\sqrt{2}}{2}\right)=2\left(\frac{\sqrt{2}}{2}\right)\cdot e^{-\left(\frac{\sqrt{2}}{2}\right)^2}=\sqrt{2}e^{-1/2}=\frac{\sqrt{2}}{\sqrt{e}}=\sqrt{\frac{2}{e}}\approx 0.858$ units^2

Bonus:

Since $k=\frac{4}{3}$ then $y=\frac{4}{3x}+\ln\left(x^2\right)$ which intersects the normal line $y=-\frac{3}{2}x+\frac{c}{2}$ at $x=2$

$y=\frac{4}{3\cdot 2}+\ln\left(2^2\right)=\frac{2}{3}+\ln 4$; thus the point $\left(2,\frac{2}{3}+\ln 4\right)$ must be on the normal line

$\frac{2}{3}+\ln 4=-\frac{3}{2}\cdot 2+\frac{c}{2} \;\Rightarrow\; \frac{c}{2}=\frac{2}{3}+\ln 4+3 \;\Rightarrow\; c=\frac{4}{3}+6+2\ln 4 \;\Rightarrow\; c=\frac{22}{3}+\ln 16$