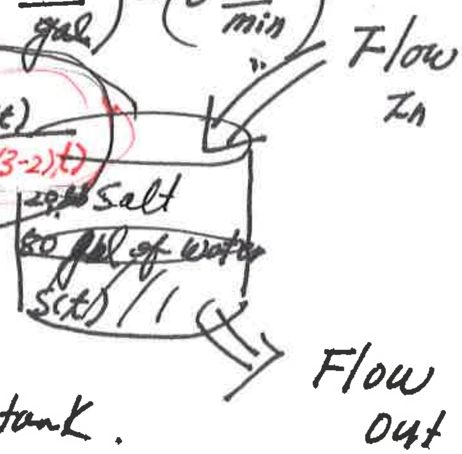
IB Math HL1 : Differential Equations Name: ____________________

Applications Problems: WS #3

1. A dilution Problem:
A tank contains 20 lb of salt dissolved in 50 gal of water. Suppose 3 gal of brine containing 2 lb of dissolved salt per gallon runs into the tank every minute and that the mixture (kept uniform by stirring) runs out of the tank at the rate of 2 gal/min. Find the amount of salt in the tank at any time t. How much salt is in the tank at the end of hour?

- *The amount of salt in the tank at the end of t minutes:* *S(t)*
- *The rate of amount of salt flow in (lb/min):* $\left(3\,\frac{gal}{min}\right)\left(\frac{2\,lb}{gal}\right)=\left(6\,\frac{lb}{min}\right)$
- *The rate of amount of salt flow out (lm/min):* $\left(\frac{2\,gal}{min}\right)\left(\frac{S(t)}{50+(3-2)t}\right)$



1) • Rate change of amount of salt

⇒ (Concentration · Flow rate)

2) • Rate change of amount of salt in the tank.

= Flow In − Flow out.

$$\boxed{\frac{dS}{dt} = 6 - 2\left(\frac{S(t)}{50+t}\right)}$$

$$\Rightarrow \frac{dS}{dt} + \left(\frac{2}{50+t}\right)\cdot S = 6 \qquad (S' + P\cdot S = Q)$$

$$P = \frac{2}{50+t} \Rightarrow I = e^{\int \left(\frac{2}{50+t}\right)dt} = e^{2\ln(50+t)} = (50+t)^2$$

$$(S')(50+t)^2 + \left(\frac{2}{50+t}\right)(50+t)^2\cdot S = 6(50+t)^2$$

$$\Rightarrow \int \frac{d}{dt}\left(S\cdot(50+t)^2\right) = \int 6(50+t)^2\cdot dt$$

$$S(50+t)^2 = \frac{6}{3}^{2}(50+t)^3 + C$$

$$\Rightarrow S = \frac{1}{(50+t)^2}\left[2(50+t)^3 + C\right]$$

$$\boxed{S(t) = 2(50+t) + C(50+t)^{-2}}$$

$t = 0 \quad S = 20\,lb.$

$$20 = \cancel{2}\cdot 50 + C(50)^{-2}$$
$$-100 \qquad -100$$

$$C = (50)^2(-80)$$

$$\Rightarrow \boxed{S(t) = 2(50+t) - (80)(50)^2(50+t)^{-2}}$$

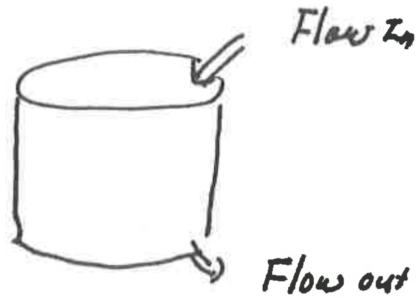
$t = 60\,min$

$$S(60) = 2(50+60) - (80)(50)^2(50+60)^{-2}$$
$$\approx \boxed{203\ lb}.$$

Exit Question

When your group members finish the following problem, get a stamp from Mrs. Shim

A tank contains 10 lb of salt dissolved in 30 gal of water. Suppose 2 gal of brine containing 1 lb of dissolved salt per gallon runs into the tank every minutes and that the mixture (kept uniform by stirring) runs out of the tank at the rate of 1.5 gal/ min. How long does it take (to nearest second) for the tank to contain 15 lb of salt?



Rate change of amount of Salt in the tank.

$\Rightarrow$ Flow In $-$ Flow out

$$\text{Flow In} = \left(2\ \frac{gal}{min}\right)\left(\frac{1\,lb}{gal}\right) = 2\,\frac{lb}{min}$$

$$\text{Flow out} = \left(1.5\ \frac{gal}{min}\right)\left(\frac{S}{30+0.5t}\right)\frac{lb}{min}$$

$$\frac{dS}{dt} = 2 - \frac{(1.5S)\cdot 2}{(30+0.5t)\cdot 2} = 2 - \frac{3S}{60+t}.$$

$$S' + \frac{3}{60+t}\cdot S = 2. \quad \Rightarrow \quad I = e^{\int\left(\frac{3}{60+t}\right)dt} = (60+t)^3$$

$$(60+t)^3\cdot S' + \left(\frac{3}{60+t}\right)(60+t)^3\cdot S = 2(60+t)^3$$

$$\Rightarrow \int d\left[S\cdot(60+t)^3\right] = \int 2(60+t)^3\,dt$$

$$S = \frac{1}{(60+t)^3}\left[\frac{1}{2}(60+t)^4 + c\right] = \frac{1}{2}(60+t) + c(60+t)^{-3}$$

$\Rightarrow$ To Find c $\Rightarrow$ $10 = \left(\frac{1}{2}\right)(60) + c(60)^{-3} \Rightarrow c = (-20)(60)^3$

$$S(t) = \frac{1}{2}(60+t) - (20)(60)^3(60+t)^{-3}$$

Solve by use of Graphing cal.

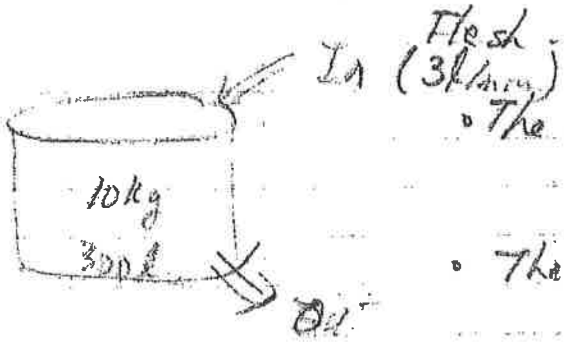
$$\Rightarrow 15 = \frac{1}{2}(60+t) - (20)(60)^3(60+t)^{-3} \Rightarrow t \approx 3\text{ min } 35\text{ sec}$$

#1. W.S. #3

Solutions.

#2.

(a)



· The amount of salt at time t: x

· The rate of amount of salt flow in: 0

· The rate of amount of salt flow out: $\left(3\,\frac{l}{min}\right)\left(\frac{x}{300}\right)$.

(i) $\frac{x}{300}\ \frac{kg}{l}$.

(ii) $\left(3\,\frac{l}{min}\right)\left(\frac{x}{300}\ \frac{kg}{l}\right) = \left(\frac{x}{100}\ \frac{kg}{min}\right)$

(iii) $\boxed{\frac{dx}{dt} = In - Out = -\frac{x}{100}}$ $(t=0,\ x=10kg)$

(iv) $\ln x = -\frac{1}{100}t + C \qquad C = \ln 10$

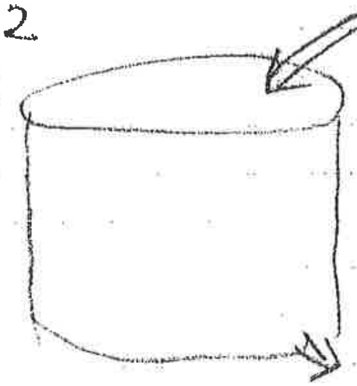
$\ln x = -\frac{1}{100}t + \ln 10 \qquad x = 10e^{-\frac{1}{100}t},\ t \geq 0$

· 40% of 10kg = 4kg $\Rightarrow 4 = 10e^{-\frac{1}{100}t}$

$t = 91.63$ minutes

2

(b)



Salt $0.2\,\frac{kg}{l}$

Out

- The rate of amount of salt flow in

$$\left(\frac{3l}{min}\right)\left(0.2\frac{kg}{l}\right) = \boxed{0.6\,\frac{kg}{min}}$$

- The rate of amount salt flow out

$$\left(3\frac{l}{min}\right)\left(\frac{x}{300}\right) = \frac{x}{100}\,\frac{kg}{min}$$

(i) $\boxed{\frac{dx}{dt} = 0.6 - \frac{x}{100} \quad (x(0) = 10)}$

$$\Rightarrow \frac{dx}{dt} = \frac{60-x}{100}$$

(ii) $\Rightarrow \frac{dx}{60-x} = \frac{dt}{100} \Rightarrow -\ln|60-x| = \frac{t}{100} + C \qquad x=10 \;\; t=0$

$$C = -\ln 50$$

$$\Rightarrow \frac{t}{100} = \ln\left(\frac{50}{60-x}\right) \Rightarrow \boxed{x = 60 - 50e^{-\frac{1}{100}t}}$$

$$\Rightarrow \frac{50}{60-x} = e^{\frac{t}{100}} \Rightarrow 50 = e^{\frac{t}{100}}(60-x)$$

When $t = 120$ minutes $\quad x = 60 - 50\cdot e^{-\frac{1}{100}(120)} \approx \boxed{44.94\,kg}$

(c) The Rate of amount salt flow in

$$\left(2\frac{l}{min}\right)\left(0.2\frac{kg}{l}\right) = 0.4\,\frac{kg}{l}$$

The Rate of amount Salt flow out

$$\left(3\frac{l}{min}\right)\left(\frac{x}{300+(3-2)t}\right) = \frac{3x}{300+t}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = 0.4 - \frac{3x}{300+t} \quad \left(x(0) = 10\right)}$$