IB Math HL2 Euler's Method:

$y_{n+1} = y_n + h \times f(x_n, y_n)$ and $x_{n+1} = x_n + h$ where h is a step size.

Example) Consider the differential Equation $\frac{dy}{dx} + \frac{xy}{4-x^2} = 1$, where $|x| < 1$ and $y(0) = 1$.

a) Use Euler's method with $h = 0.25$ to find an approximate value of y when x=1. Give the final answer to 3 significant figures.

x	y	dy
0	1	0.25
0.25	1.25	.2301
0.5	1.4801	.2006
0.75	1.6807	.1583
1	1.84	

$\Delta x = 0.25 \approx dx$.

$\Delta y \approx dy$.

$dy = \left(1 - \frac{xy}{4-x^2}\right) dx$

$= \left(1 - \frac{\cancel{0 \cdot 1}}{4-0^2}\right) 0.25$

$= 0.25$

b) Solve algebraically and compare the answer with the answer (a).

$\frac{dy}{dx} + \frac{xy}{4-x^2} = 1$

$y' + p \cdot y = Q$.

$p = \frac{x}{4-x^2} \Rightarrow I = e^{\int \frac{x}{4-x^2} dx}$

$u = 4 - x^2$

$-\frac{1}{2} du = x\,dx$

$I = e^{-\frac{1}{2}\ln(4-x^2)}$

$= (4-x^2)^{-\frac{1}{2}}$.

$\frac{1}{\sqrt{4-x^2}} \cdot y' + \frac{x}{(4-x^2)^{3/2}} \cdot = \frac{1}{\sqrt{4-x^2}}$

$\int \frac{d}{dx}\left[y \cdot \frac{1}{\sqrt{4-x^2}}\right] = \int \frac{1}{\sqrt{4-x^2}} dx$

$y = \sqrt{4-x^2}\left[\arcsin\left(\frac{x}{2}\right) + c\right]$

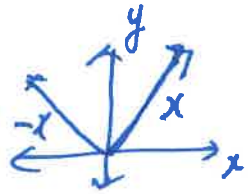
$y(0) = 1$

$1 = \sqrt{4}[0 + c] \Rightarrow c = \frac{1}{2}$

$y = \sqrt{4-x^2}\left[\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\right]$

$\boxed{y(1) = 1.77}$.

IB Math HL2 : Differentiability



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Example 1)

Given $f(x) = 2x + |x|$:

a) Prove that f is continuous but not differentiable at the point (0, 0)

$$f(x) = \begin{cases} 3x, & x \ge 0 \\ x, & x < 0 \end{cases}$$

1) Continuity : $\left(\lim_{x \to 0^+} 3x = \lim_{x \to 0^-} x \right)$

$0 = 0$

Continuous

2) Differentiability $\lim_{x \to 0^+} (3x)' \neq \lim_{x \to 0^-} (x)'$

$3 \neq 1$

Not differentiable

b) Determine the value of $\int_{-a}^{a} f(x)dx$ where $a > 0$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} x\,dx + \int_{0}^{a} 3x\,dx$$

$$= \frac{1}{2}x^2\Big]_{-a}^{0} + \frac{3}{2}x^2\Big]_{0}^{a}$$

$$= -\frac{1}{2}a^2 + \frac{3}{2}a^2 = \boxed{a^2}$$

Example 2)

1. Given $f(x) = \begin{cases} \sin(x-1) + cx & x \le 1 \\ x^2 - x + d & x > 1 \end{cases}$:

Find constants $c, d \in R$. So that the function is differentiable for all x.

1) $\lim_{x \to 1^+} (x^2 - x + d) = \lim_{x \to 1^-} \sin(x-1) + cx$

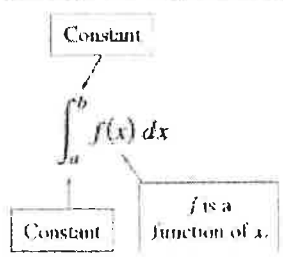
$1 - 1 + d = c \Rightarrow \boxed{d = c}$ $\boxed{d = 0}$

2) $\lim_{x \to 1^+} 2x - 1 = \lim_{x \to 1^-} \cos(x-1) + c$

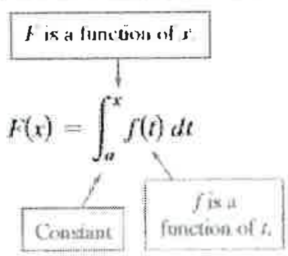
$1 = \cos(0) + c \Rightarrow \boxed{c = 0}$

Second Fundamental Theorem:

The Definite Integral as a Number



The Definite Integral as a Function of x



Given $F(x) = \int_0^x \cos t\,dt$ $\Rightarrow$ $\sin x$

a) Predict the function of $F(x)$. And then validate your prediction.

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \boxed{\sin x}$$

b) Predict $\frac{d}{dx}[F(x)] = \frac{d}{dx}\int_0^x \cos t\,dt$. And then validate your prediction.

$$\frac{d}{dx}[\sin x] = \boxed{\cos x}$$

The Second Fundamental Theorem: $\frac{d}{dx}\int_a^x f(t)\,dt = \boxed{f(x)}$ $\quad a \in \mathbb{R}$

Example 1) Using the second fundamental theorem, Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right]$ $= \boxed{\sqrt{x^2+1}}$

Chain Rule.

$$\frac{d}{du}\cdot\frac{du}{dx} = \frac{d}{dx}$$

Example 2) Find the derivative of $F(x) = \left[\int_{\pi/2}^{x^2} (\cos t)\,dt\right]$ $\quad \frac{dF}{dx}$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$\frac{d}{dx}\left[\int_{\frac{\pi}{2}}^{x^2} \cos t\,dt\right] = \frac{d}{du}\left[\int_{\frac{\pi}{2}}^{u} \cos t\,dt\right]\cdot\left[\frac{du}{dx}\right] = [\cos x^2]\,2x$$

Example 3) Find $F'(x)$ where $F(x) = \left[\int_x^{x+5} (\sec^4 t)\,dt\right]$ $\quad \frac{dF}{dx}$ $\Rightarrow \frac{d}{dx}\left[\int_x^{x+5} \sec^4 t\,dt\right]$

$$= \frac{d}{dx}\left[\int_x^0 \sec^4 t\,dt + \int_0^{x+5} \sec^4 t\,dt\right]$$