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IB Math HL2: Euler's From and De Moivre's Theorem

$z = a + bi \Rightarrow r\,cis\,\theta \Rightarrow re^{i\theta}$

$r = |z| = \sqrt{a^2+b^2}$, $\arg(z) = \theta$.

Complex Number can be written as Euler's Form: $rcis\theta = re^{i\theta}$

Show the equality of $rcis\theta = re^{i\theta}$ using the series below:

Maclaurin Series:

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.........
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.........

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$........

$cis x = \cos + i \sin = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)$

$x = \theta$.

$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} - i\frac{x^5}{5!} \cdots$

$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix^5)}{5!} \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{5!} \cdots$

Examples)

$\therefore cis x = e^{ix}$ $(cis\theta = e^{i\theta})$

a) Write $e^{-i\frac{\pi}{4}}$ in Cartesian form

$e^{-i\frac{\pi}{4}} = e^{i(-\frac{\pi}{4})} = cis(-\frac{\pi}{4})$

$(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

b) Write i^{-i} in Euler's form.

$(i)^{-i} = (cis(\frac{\pi}{2}))^{-i} = (e^{i(\frac{\pi}{2})})^{-i} = e^{-i^2(\frac{\pi}{2})}$

$= e^{\frac{\pi}{2}}$

Example) Using $cis\theta = e^{i\theta}$, prove

a) $cis\theta cis\beta = cis(\theta+\beta)$

$cis\theta \cdot cis\beta$

$= e^{i\theta} \cdot e^{i\beta} = e^{i(\theta+\beta)} = cis(\theta+\beta)$

b) $\frac{cis\theta}{cis\beta} = cis(\theta-\beta)$

$\frac{cis\theta}{cis\beta} = \frac{e^{i\theta}}{e^{i\beta}} = e^{i(\theta-\beta)}$

$= cis(\theta-\beta)$

- De Moivre's Theorem: $[r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$ $\Rightarrow (r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,n\theta.$

✱ Validate $[r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$ for $n \in Z^+$ by Math Induction

1) when $n = 1$ $\Rightarrow \underbrace{r(\cos\theta + i\sin\theta) = r^1(\cos\theta + i\sin\theta)}^{\text{DMT}}$

DMT is true for $n = 1$.

2) when $n = k$ $\quad k \in Z^+$ Assume DMT is true.

$[r(\cos\theta + \sin\theta)]^k = r^k[\cos k\theta + i\sin k\theta]$

3) If $n = k+1$.

$[r(\cos\theta + i\sin\theta)]^{k+1} = r\cdot\boxed{r^k[\cos\theta + i\sin\theta]^k}(\cos\theta + i\sin\theta)$

$= r\cdot r^k(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= r^{k+1}[\cos k\theta\cdot\cos\theta + i\cos k\theta\cdot\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta]$

$= r^{k+1}[(\cos\underset{A}{k\theta}\cos\underset{B}{\theta} - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$

using compound Angle identities

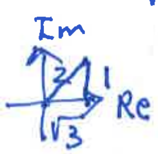
$= r^{k+1}(\cos(k\theta + \theta) + i\sin(k\theta + \theta))$

$= r^{k+1}(\cos\theta(k+1) + i\sin\theta(k+1))$

4) $\therefore$ DMT is true for $n \in Z^+$.

Example 1) Find the exact value of $(\sqrt{3} + i)^8$.

$\sqrt{3} + i \Rightarrow r = \sqrt{3+1} = 2.$



$\theta = \frac{\pi}{6}$

$\sqrt{3} + i = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

$\Rightarrow (\sqrt{3} + i)^8 = \left[2\,\text{cis}\left(\frac{\pi}{6}\right)\right]^8 = 2^8\,\text{cis}\left(8^4\cdot\frac{\pi}{6_3}\right) = 2^8\,\text{cis}\left(\frac{4\pi}{3}\right)$

$= 2^8\left[\left(-\frac{1}{2}\right) + i\left(-\frac{\sqrt{3}}{2}\right)\right]$

$= \boxed{-2^7 - i\,2^7\cdot\sqrt{3}}$

- nth roots of $r(\cos\theta + i\sin\theta)$ using De Moivre's Theorem

$$(a+bi)^{\frac{1}{n}} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}[\cos(\frac{\theta+2k\pi}{n}) + i\sin(\frac{\theta+2k\pi}{n})] \quad \Rightarrow (r\,\text{cis}\,\theta)^{\frac{1}{n}}$$

$(a+bi)^{-n}$

where $k = 0, 1, 2, \ldots, n-1$

$= r^{\frac{1}{n}} \cdot \text{cis}(\frac{\theta}{n})$

$\cdot\ \text{cis}\,\theta = \text{cis}(\theta + 2n\pi)$

Example 2) Given $z^4 - 81 = 0$,

a) Solve the equation by factorization.

$z^4 - 81 = 0$

$(z^2+9)(z^2-9) = 0$

$(z^2+9)(z+3)(z-3) = 0$

$z^2 = -9$

$z = \pm 3i$, $z = -3$, $z = 3$.

b) Solve the equation by De Moivres' theorem. $\quad 81 = 81\,\text{cis}[0+2k\pi]$.

$z^4 - 81 \Rightarrow z^4 = 81 = 81\,\text{cis}[0+2k\pi]. \Rightarrow z = [81\,\text{cis}(0+2k\pi)]^{\frac{1}{4}}$

DMT. $z = (81)^{\frac{1}{4}}[\text{cis}(\frac{0+2k\pi}{4})]$.

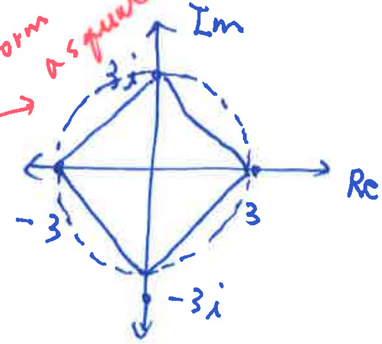
$k=0 \quad z = 3\,\text{cis}(0) = 3(\cos 0 + i\sin 0) = 3$

$k=1 \quad z = 3[\cos(\cancel{\frac{2\pi}{4}})^{0} + i\,\cancel{\sin}(\frac{2\pi}{4})^{1}] = 3i$

$k=2 \quad z = 3[\cos(\pi) + i\sin(\pi)] = -3$

$k=3 \quad z = 3[\cos(\frac{6\pi}{4}) + i\sin(\frac{6\pi}{4})] = -3i$

Form a square.

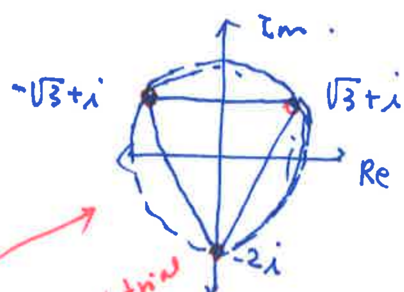


Example 3) Find the cubic roots of $8i$

$8i = 8\,\text{cis}(\frac{\pi}{2} + 2\pi k)$

$\Rightarrow [8\,\text{cis}(\frac{\pi}{2} + 2\pi k)]^{\frac{1}{3}}$

$= 8^{\frac{1}{3}}\,\text{cis}[\frac{\pi}{6} + \frac{2k\pi}{3}]$



$k=0 \Rightarrow 2[\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})] = \sqrt{3}+i$

$k=1 \Rightarrow 2[\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})] = -\sqrt{3}+i$

$k=2 \Rightarrow 2[\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})] = -2i$

Form an equilateral Triangle