Name: ________________ Group Members: ________________________________

Exploration 7-2a: Differential Equation for Compound Interest

Date: __________

Objective: Write and solve a differential equation for the amount of money in a savings account as a function of time.

When money is left in a savings account, it earns interest equal to a certain percent of what is there. The more money you have there, the faster it grows. If the interest is *compounded continuously,* the interest is added to the account the instant it is earned.

1. For continuously compounded interest, the instantaneous rate of change of money is directly proportional to the amount of money. Define variables for time and money, and write a differential equation expressing this fact.

M: The amount of money at time t

t: Time (yrs). $\boxed{\frac{dM}{dt} = kM}$

2. Separate the variables in the differential equation in Problem 1, then integrate both sides with respect to *t*. Transform the integrated equation so that the amount of money is expressed explicitly in terms of time.

$\frac{dM}{M} = k\,dt \Rightarrow \int \frac{dM}{M} = \int k\,dt$

$\ln|M| = kt + C \qquad M = e^{kt} \cdot e^{C}$

3. The integrated equation from Problem 2 will contain *e* raised to a power containing *two* terms. Write this power as a product of two different powers of *e*, one that contains the time variable and one that contains no variable.

$M = e^{kt} \cdot A$

where $e^{C} = A$

4. You should have the expression e^{C} in your answer to Problem 3. Explain why e^{C} is always positive.

A positive number raised to any power is positive.

5. Replace e^{C} with a new constant, C_1. If C_1 is allowed to be positive or negative, explain why you no longer need the ± sign that appeared when you removed the absolute value in Problem 2.

$M > 0$ Let $A = e^{c}$ positive

$M < 0$ Let $A = e^{c}$ negative

6. Suppose that the amount of money is $1000 when time equals zero. Use this initial condition to evaluate C_1.

$M(0) = A e^{k \cdot 0} = 1000$

7. If the interest rate is 5% per year, then d(money)/d(time) = 0.05(money), in dollars per year. What, then, does the proportionality constant in Problem 1 equal?

$k = 0.05$

8. How much money will be in the account after 1 year? 5 years? 10 years? 50 years? 100 years? Do the computations in the most time-efficient manner.

$M(5) = \$1284.03$ $\qquad M(100)$

$M(10) \approx \$1648.72$ $\qquad \approx \$148,413..$

$M(50) \approx \$12,182.49$

9. How long would it take for the amount of money to double its initial value?

$t = \frac{\ln 2}{0.05} \approx 13.8629$ yr.

$2A = A e^{kt}$

10. What did you learn as a result of doing this Exploration that you did not know before?

ne: ______________ Group Members: ______________________________

xploration 7-3a: Differential Equation r Memory Retention

Date: __________

jective: Write and solve a differential equation for the number of names remembered a function of time.

Member is a freshman at a large university. One ning he attends a reception at which there are many mbers of his class whom he has not met. He wants to dict how many new names he will remember at the end he reception.

. Ira assumes that he meets people at a constant rate of R people per hour. Unfortunately, he forgets names at a rate proportional to y, the number he remembers. The more he remembers, the faster he forgets! Let t be the number of hours he has been at the reception. What does dy/dt equal? (Use the letter k for the proportionality constant.)

t: time
R: constant (people he meets)
y: #s of names.

$$\frac{dy}{dt} = R - ky$$

2. The equation in Problem 1 is a differential equation because it has differentials in it. By algebra, separate the variables so that all terms containing y appear on one side of the equation and all terms containing t appear on the other side.

$$\int (R-ky)^{-1}\,dy = \int dt$$

$$-\frac{1}{k}\ln(R-ky) = t + C$$

$$\ln|R-ky| = -kt + C_1$$

3. Integrate both sides of the equation in Problem 2. You should be able to make the integral of the reciprocal function appear on the side containing y.

4. Show that the solution in Problem 3 can be transformed into the form

$$ky = R - Ce^{-kt}$$

where C is a constant related to the constant of integration. Explain what happens to the absolute value sign that you got from integrating the reciprocal function.

$$R - ky = e^{-kt + C_1}$$
$$= e^{-kt} \cdot e^{C_1}$$
$$= e^{-kt} \cdot A$$
$$ky = R - Ae^{-kt}$$

5. Use the initial condition $y = 0$ when $t = 0$ to evaluate the constant C.

$$k \cdot 0 = R - Ae^{0} \Rightarrow A = R$$
$$\Rightarrow y = \frac{R}{k}(1 - e^{-kt})$$

6. Suppose that Ira meets 100 people per hour, and that he forgets at a rate of 4 names per hour when $y = 10$ names. Write the particular equation expressing y in terms of t.

ky

$R = 100$ $\quad$ $ky = 4$ when $y = 10$

$k = \frac{4}{y} = \frac{4}{10}$

$k = 0.4$

$$\boxed{y = 250(1 - e^{-0.4t})}$$

7. How many [illegible] will Ira have remembered at the end of the reception, $t = 3$ h?

$y(3) \approx 175$ names (174)

8. What did you learn as a result of doing this Exploration that you did not know before?