Review – Algebra and Functions & Equations (10 questions)

♦ **Paper 1 Review** ♦ No calculator allowed [**worked solutions** included]

1. Find the set of values of x for which $\left(e^x - 2\right)\left(e^x - 3\right) \le 2e^x$.

2. Given that $3x^2 - kx + 12$ is positive for all values of x, find the range of possible values for k.

3. Find the value(s) of m so that the equation $mx^2 - mx + 1 = 0$ has exactly one real root.

4. Find all real solutions for the equation $x = \sqrt{x+5} - 3$.

5. Solve the inequality $2\left|x+3\right| \le x + 15$.

6. Find the range of values of k for which $y = 2x + k$ and $x^2 + y^2 = 4$ do no intersect.

7. Find the **exact** solution(s) to the equation $8e^2 - 2e\ln x = (\ln x)^2$.

8. Find the quadratic equation having the roots $1+5i$ and $1-5i$.

9. One root of the equation $x^2 + ax + b = 0$, where a and b are real constants, is $2+3i$. Find the value of a and the value of b.

10. Find the square roots of $3+4i$.

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Worked Solutions

[**note:** although these are non-calculator questions some of the answers have been confirmed using a TI-84 GDC – which is good practice because it's possible that these questions could be on Paper 2]

1. Find the set of values of x for which $\left(e^x-2\right)\left(e^x-3\right)\le 2e^x$.

The left side is easily expand … $\left(e^x-2\right)\left(e^x-3\right)=e^{2x}-5e^x+6$ … Now, make the right side zero – and can already see that the final expression on the left side will factor quite nicely.

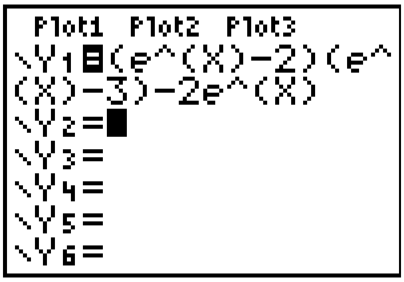
$e^{2x}-7e^x+6\le 0 \quad \Rightarrow \quad \left(e^x-6\right)\left(e^x-1\right)\le 0$ … use a 'sign chart'

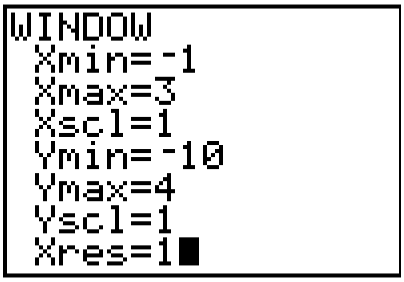
when does $e^x-6=0$? $\Rightarrow$ $x=\ln 6\approx 1.79$ when does $e^x-1=0$? $\Rightarrow$ $x=0$

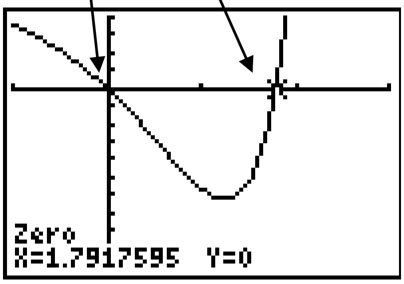
		0		ln 6	
e^x-6	neg.		neg.		pos.
e^x-1	neg.		pos.		pos.
$\left(e^x-6\right)\left(e^x-1\right)$	pos.		neg.		pos.

Therefore, solution set for the inequality is $0\le x\le\ln 6$ (exactly) **or** $0\le x\le 1.79$ (approximately)

Graph on GDC to confirm:







2. Given that $3x^2-kx+12$ is positive for all values of x, find the range of possible values for k.

Since the leading coefficient of this quadratic is positive (i.e. 3), then its corresponding equation in two variables, $y=3x^2-kx+12$, is a parabola that opens up. If it is always positive, then it does not touch the x-axis – which also means that it has **no real zeros**. A quadratic equation will have no real zeros if the **discriminant is negative** $\left(\text{i.e. } b^2-4ac<0\right)$. Hence, find the values of k that satisfy the inequality

$k^2-4(3)(12)<0 \quad \Rightarrow \quad k^2-144<0$.

$k^2-144<0 \Rightarrow (k+12)(k-12)<0$ To solve inequality, only need to test three values for x …one less than -12, one between -12 and 12, and one greater than 12.

This shows that the solution set for k is $-12<k<12$

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3. Find the value(s) of m so that the equation $mx^2 - mx + 1 = 0$ has exactly one real root.

The quadratic equation will have exactly one real root when the discriminant is zero.

$$m^2 - 4(m)(1) = m^2 - 4m = m(m-4) = 0 \quad \Rightarrow \quad \text{Either } m = 0 \text{ or } m = 4$$

But, if $m = 0$, then equation is $1 = 0$ which is a false statement. Hence, only solution is $m = 4$

4. Find all real solutions for the equation $x = \sqrt{x+5} - 3$

$$x + 3 = \sqrt{x+5}$$
$$(x+3)^2 = \left(\sqrt{x+5}\right)^2$$
$$x^2 + 6x + 9 = x + 5$$
$$x^2 + 5x + 4 = 0$$
$$(x+1)(x+4) = 0$$

Hence, $x = -1$ or $x = -4$

Whenever, squaring both sides in solving an equation one must check the solutions because extraneous solutions may have been introduced.

Check $x = -1$: $-1 = \sqrt{-1+5} - 3 \Rightarrow -1 = \sqrt{4} - 3 \Rightarrow -1 = 2 - 3$ **<u>OK</u>**

Check $x = -4$: $-4 = \sqrt{-4+5} - 3 \Rightarrow -4 = \sqrt{1} - 3 \Rightarrow -4 \neq 1 - 3$ **<u>Not OK</u>**

Therefore, only solution is $x = -1$

5. Solve the inequality $2|x+3| \leq x + 15$

$$|x+3| \leq \frac{x}{2} + \frac{15}{2}$$

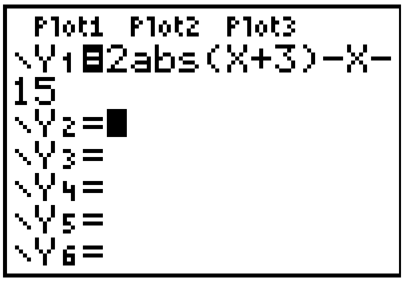
$x + 3 \leq \frac{x}{2} + \frac{15}{2}$ and $-(x+3) \leq \frac{x}{2} + \frac{15}{2}$

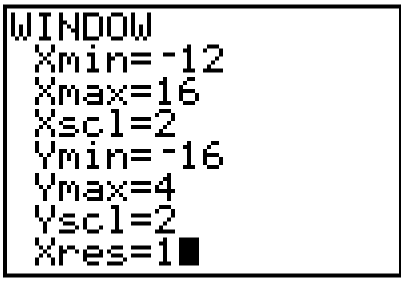
$\frac{x}{2} \leq \frac{9}{2}$ and $\frac{3x}{2} \geq -\frac{21}{2}$

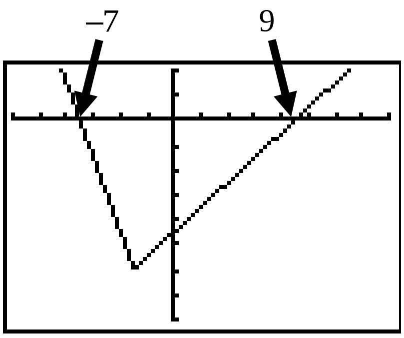
$x \leq 9$ and $x \geq -7$

$$-7 \leq x \leq 9$$

confirm on GDC:







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6. Find the range of values of k for which $y = 2x + k$ and $x^2 + y^2 = 4$ do no intersect.

Substitute $2x+k$ in for y in second equation $\Rightarrow$ $x^2 + (2x+k)^2 = 4$ $\Rightarrow$ $x^2 + 4x^2 + 4kx + k^2 - 4 = 0$

$5x^2 + 4kx + k^2 - 4 = 0$ If the two equations do not intersect, this equation has no solution. This is equivalent to the equation $y = 5x^2 + 4kx + k^2 - 4$ having no real zeros $\Rightarrow$ discriminant is negative

$(4k)^2 - 4(5)(k^2-4) < 0$ $\Rightarrow$ $-4k^2 + 80 < 0$ $\Rightarrow$ $k^2 - 20 > 0$ note: $\sqrt{20} = 2\sqrt{5}$

$(k + 2\sqrt{5})(k - 2\sqrt{5}) > 0$ check k less than $-2\sqrt{5}$, between $-2\sqrt{5}$ and $2\sqrt{5}$, and greater than $2\sqrt{5}$

This leads to the following solution set for k: $k < -2\sqrt{5}$ or $k > 2\sqrt{5}$ (exactly)

7. Find the exact solution(s) to the equation $8e^2 - 2e\ln x = (\ln x)^2$

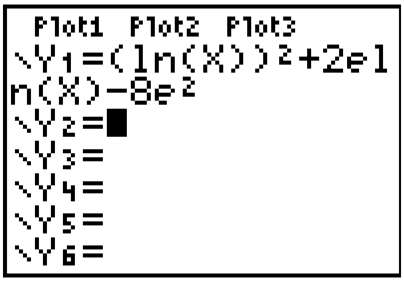
$(\ln x)^2 + 2e\ln x - 8e^2 = 0$ Let $y = \ln x$ $\Rightarrow$ $y^2 + 2ey - 8e^2 = 0$ quadratic formula gives …

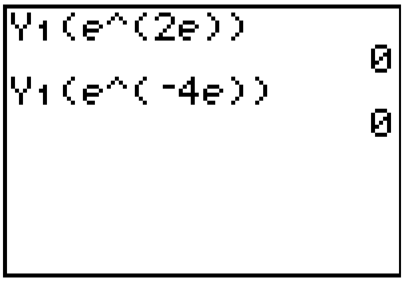
$$y = \frac{-2e \pm \sqrt{(2e)^2 - 4(1)(-8e^2)}}{2} = \frac{-2e \pm \sqrt{4e^2 + 32e^2}}{2} = \frac{-2e \pm \sqrt{36e^2}}{2} = \frac{-2e \pm 6e}{2} \quad y = 2e \text{ or } y = -4e$$

for $y = 2e$ $\Rightarrow$ $\ln x = 2e$ $\Rightarrow$ $x = e^{2e}$

for $y = -4e$ $\Rightarrow$ $\ln x = -4e$ $\Rightarrow$ $x = e^{-4e}$ exact solutions are $x = e^{2e}$ or $x = e^{-4e}$

confirm on calculator:





8. Find the quadratic equation having the roots $1+5i$ and $1-5i$

If $x = 1+5i$ and $x = 1-5i$ are roots, then $x-(1+5i)$ and $x-(1-5i)$ are factors of the equation

$[x-(1+5i)][x-(1-5i)] = [x-1-5i][x-1+5i] = [(x-1)-5i][(x-1)+5i] = (x-1)^2 - (5i)^2 =$

$= x^2 - 2x + 1 + 25 = x^2 - 2x + 26$ quadratic equation with roots $1+5i$ and $1-5i$ is $x^2 - 2x + 26 = 0$

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9. One root of the equation $x^2 + ax + b = 0$, where *a* and *b* are real constants, is $2+3i$.
Find the value of *a* and the value of *b*.

The other root must be the conjugate of $2+3i$, which is $2-3i$. If these are the roots, then the factors must be $x-(2+3i)$ and $x-(2-3i)$

$$\left[x-(2+3i)\right]\left[x-(2-3i)\right]=\left[(x-2)-3i\right]\left[(x-2)+3i\right]=(x-2)^2-(3i)^2=x^2-4x+4+9$$

The quadratic with these roots is $x^2-4x+13=0$, therefore, $a=-4$ and $b=13$

10. Find the square roots of $3+4i$

Remember…every complex number will have **<u>two</u>** square roots

If $x+yi$ is the square root of $3+4i$, then $(x+yi)^2=3+4i$ Expand $(x+yi)^2$

$$x^2+2xyi+y^2i^2=3+4i$$
$$x^2-y^2+2xyi=3+4i$$

Now equating the real parts and the imaginary parts from both sides of the equation gives

$x^2-y^2=3$ and $2xy=4$ It follows from the 2nd equation that $y=\dfrac{4}{2x}=\dfrac{2}{x}$

Substituting gives $x^2-\left(\dfrac{2}{x}\right)^2=3 \Rightarrow x^2-\dfrac{4}{x^2}-3=0$ … multiplying both sides by x^2

$x^4-3x^2-4=0 \Rightarrow (x^2-4)(x^2+1)=0 \Rightarrow (x+2)(x-2)(x^2+1)=0$

Then $x=-2$ or $x=2$

If $x=-2$, then $y=-1$ … and if $x=2$, then $y=1$

Therefore, the two square roots of $3+4i$ are: $-2-i$ and $2+i$

confirm on GDC: