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IB Math HL2: Geometric Power Series **Name:** ____________ **Period:** _______

Estimation of f(a) using Series.
First Example)

1) Integrate the geometric series $\sum_{n=0}^{\infty} x^n$ $= 1 + x + x^2 + x^3 \ldots$ $r = |x| < 1$

$$S_\infty = \int \frac{1}{1-x}\,dx = \boxed{\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}} \ldots \quad -1 < x < 1$$

2) Hence, show $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x)$

$$\Rightarrow \boxed{-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}} = \ln(1-x)^{-1} = \ln\left[\frac{1}{1-x}\right].$$

3) Estimate $\ln(1.5)$ using first 4 none zero terms of series.

$$\frac{1}{1-x} = 1.5 = \frac{3}{2} \qquad \Rightarrow \ln\left[\frac{1}{1-\frac{1}{3}}\right] = \ln(1.5)$$

$$2 = 3 - 3x \Rightarrow 3x = 1$$

$$\boxed{x = \frac{1}{3}} \Leftarrow \boxed{-1 < x < 1}$$

$$= \frac{1}{3} + \frac{(\frac{1}{3})^2}{2} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^4}{4}.$$

$$= \frac{1}{3} + \frac{1}{18} + \frac{1}{81} + \frac{1}{81 \cdot 4}.$$

$$\approx 0.404321.$$

Second Example)

1) Express $f(x) = \frac{1}{1+x^2}$ as alternating geometric series.

$$S_\infty = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \boxed{\sum_{n=0}^{\infty} (-1)^n (x^2)^n}$$

$$|-x^2| < 1 \Rightarrow \boxed{-1 < x < 1}$$

2) Find the series for $\int \frac{1}{1+x^2}\,dx$

$$\int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n (x^2)^n \right) \Rightarrow \boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n+1})}{2n+1}}$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^{2n}).$$

$+C$

3) Hence, show $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$

$$x + \frac{(-1)^1 x^3}{3} + \underbrace{\frac{(-1)^2 (x)^5}{5}}_{k=2} + \underbrace{\frac{(-1)^3 (x)^7}{7}}_{k=3} + \frac{(-1)^4 (x)^9}{9}.$$

4) Estimate π using the first 5 none zero terms of the series above.

$$x = 1 \Rightarrow \arctan(1) = \frac{\pi}{4} \Rightarrow \pi = 4\arctan(1).$$

$$\pi \approx 4\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right] \approx 3.33968$$

Third Example)

Use a power series to approximate $\int_0^1 e^{-x^2}\,dx$ with an error of less than 0.01.

Cal. [GDC]

$\int_0^1 e^{-x^2}dx \approx 0.746824.$

≈ 0.75

$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} \cdots$

$u = -x^2$

$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} \cdots$

$\int e^{-x^2}dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!}\right]_{x=0}^{x=1}$

$= \boxed{1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}} - \cancel{\frac{1}{11\cdot 5!}}$

$\approx 0.747\cancel{486}$

$\approx 0.75.$

$\frac{1}{9\cdot 4!} \approx 0.00463$

$\approx 0.005.$

Fourth Example)

Find the power series for $f(x) = \sin^2 x$.

$f(x) = \sin x \cdot \sin x.$

$= \frac{1 - \cos 2x}{2}. = \frac{1}{2} - \frac{1}{2}\cos 2x.$

$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} \cdots$

$u = 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} \cdots$

$f(x) = \sin^2 x = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{2^2\cdot x^2}{2!} + \frac{2^4\cdot x^4}{4!} - \frac{2^6\cdot x^6}{6!} \cdots\right].$

$= +\frac{2\cdot x^2}{2!} - \frac{2^3\cdot x^4}{4!} + \frac{2^5\cdot x^6}{6!} \cdots$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)^{2n-1}x^{2n}}{(2n)!}$