IB HL MATH 1 SECOND SEMESTER FINAL STUDY GUIDE

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CHAPTER 21: INTEGRATION

Standard Integrals

$$\int k dx = kx + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + c$$

$$\int \frac{1}{kx} dx = \frac{\log(kx)}{k} + c$$

$$\int \sin(kx)\, dx = -\frac{\cos(kx)}{k} + c$$

$$\int \cos(kx)\, dx = \frac{\sin(kx)}{k} + c$$

$$\int \sec^2(kx)\, dx = \frac{\tan(kx)}{k} + c$$

$$\int \sec(kx)\tan(kx)\, dx = \frac{\sec(kx)}{k} + c$$

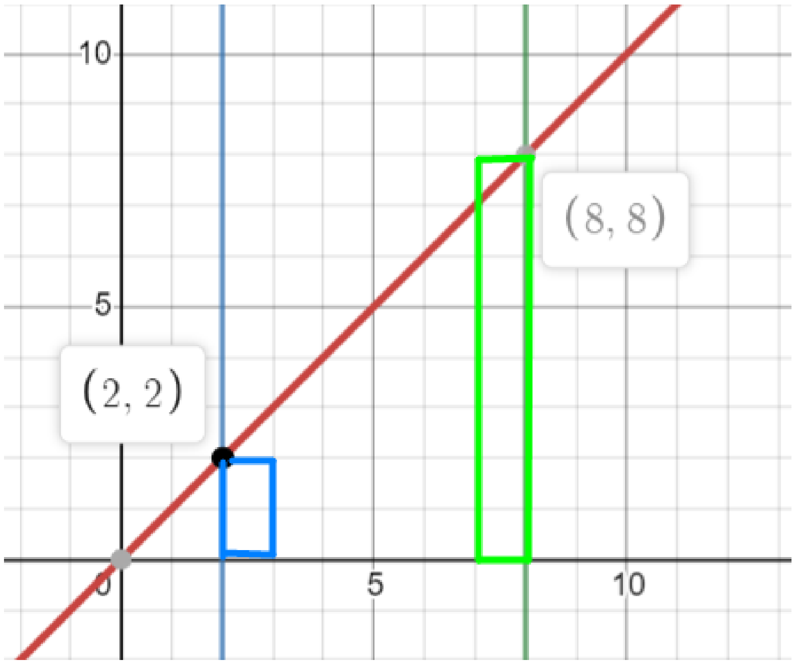
Area Under a Curve: Rectangle Method

Left Rectangles: Start with the first value on the left side of the measured graph

- Left Rectangles are the **under estimation** when the slope is generally positive
- Left Rectangles are the **over estimation** when the slop is generally negative

Right Rectangles: Start with the last value on the right side of the measured graph

- Right Rectangles are the **over estimation** when the slope is generally positive
- Right Rectangles are the **under estimation** when the slop is generally negative



In this graph $\int_2^8 x dx$, the blue is the left rectangle, as it starts on the first value (2) and branches off its y value for the given length of the rectangle. The green is the right rectangle, as it starts with the last value (8), and branches inward off its y value for the given length of the rectangle

Sigma Notation

- Left Rectangles: $\sum_0^{n-1}\left(\left(\frac{b-a}{n}\right)f\left(a+\left(\frac{b-a}{n}\right)k\right)\right)$
- Right Rectangles: $\sum_1^{n}\left(\left(\frac{b-a}{n}\right)f\left(a+\left(\frac{b-a}{n}\right)k\right)\right)$

FTC (Fundamental Theorem of Calculus)

$$\int_a^b f(x)dx = F(b) - F(a)$$

Properties of Definite Integrals

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

u-Substitution

u-Substitution with Definite Integrals

if $\int_a^b (kx+5)dx$, then the values of a and b must be plugged from a and b, therefore with u-substitution, its $\int_{a+5}^{b+5} kudu$

u-Substitution with change of variable

take the initial equation u = x and substitute an x value with u

Integration with Trig Identities

Use the booklet for information regarding trig identities

If a sin-cosine integration has an odd power $\int \sin^3(x)\cos(x)\,dx$, then u-substitution is possible

Proving $\int \sec(x)\,dx$

$$\int \frac{\sec(x)}{1}dx$$

$$\int \frac{\sec(x)}{1} \times \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx$$

$$\int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}dx$$

$$u = \sec(x)+\tan(x)\,, \qquad du = \sec(x)\tan(x)+\sec^2(x)\,dx$$

$$\int \frac{1}{u}du = \ln(u)+c$$

$$\therefore \int \sec(x)\,dx = \ln(\sec(x)+\tan(x))+c$$

Review Questions

1. $\int \left(\frac{6}{x} - \sin(7x)\right)dx$

$6\ln(x) + \frac{1}{7}(\cos(7x)) + c$

2. Suppose $y = \sec(7x)$ and $\frac{dy}{dx} = 7\sec(7x)\tan(7x)$. Hence find an **exact particular solution** for $g(x) = \sec(7x)\tan(7x)$ and $g(0) = \sqrt{7}$

$$\int 7\sec(7x)\tan(7x) = \sec(7x)+c$$

$$\int \sec(7x)\tan(7x) = \frac{\sec(7x)}{7}+c$$

$$\sqrt{7} = \frac{\sec(0)}{7}+c$$

$$c = \sqrt{7} - \frac{1}{7}$$

3. Find $\int 2x^2\sqrt{x^3+1}dx$ by u-Substitution

$$u = x^3+1, \qquad du = 3x^2dx, \qquad \frac{2}{3}du = 2x^2dx$$

$$\frac{2}{3}\int u^{\frac{1}{2}}du$$

$$\frac{2}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}}$$

$$\frac{4}{9}(x^3+1)^{\frac{3}{2}}+c$$

4. $\int_1^3 x\sqrt{2x+5}dx$

$$u=2x+5, \qquad du=2dx, \qquad \frac{1}{2}du=dx, \qquad x=\frac{u-5}{2}$$

$$\frac{1}{2}\int_1^3 u^{\frac{1}{2}}\left(\frac{u-5}{2}\right)du$$

$$\frac{1}{4}\int_1^3 (u^{\frac{3}{2}}-5u^{\frac{1}{2}})du$$

$$\frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{10}{3}u^{\frac{3}{2}}\right]_{2(1)+5=7}^{2(3)+5=11}$$

12.2

5. $\int\left(t-t\sqrt{t}\right)dt$

$$\int\left(t-t^{\frac{3}{2}}\right)dt$$

$$\frac{1}{2}t^2-\frac{2}{5}t^{\frac{3}{2}}+c$$

6. $\int\frac{5x-1}{x-3}dx$ (write in forms of $\int A+\frac{B}{x-3}dx$)

$$\int\frac{5(x-3)+14}{x-3}dx$$

$$\int 5+\frac{14}{x-3}dx$$

$$5x+14(\ln(x-3))+c$$

7. Given $f(x)=-5x^2+10$, write in sigma notation the estimated area with 10 right-hand rectangles in the interval [1,3].

$$\Delta x=\frac{3-1}{10}=\frac{1}{5}$$

$$\sum_{k=1}^{10}\frac{1}{5}f\left(1+\frac{1}{5}k\right)$$

-27.4

8. Discuss if the area of $\sum_{k=1}^{10}\frac{1}{5}f\left(1+\frac{1}{5}k\right)$ is overestimated or underestimated compared to the area found by FTC

$$\int_1^3 (-5x^2 + 10)dx$$

-23.3, The value is overestimated compared to FTC

9. $\int_1^e \frac{(\ln(x))^3}{x} dx$

$u = \ln(x)\,,\qquad du = \frac{1}{x}dx,\qquad$ and when $x = 1$ & e, $\qquad u = 0$ & 1

$$\int_0^1 u^3 du$$

$$\left[\frac{1}{4}u^4\right]\begin{matrix} u=1 \\ u=0 \end{matrix}$$

$$\frac{1}{4}$$

10. $\int \sin^3(x)\cos^2(x)\,dx$

$$\int \sin^2(x)\sin(x)\cos^2(x)\,dx$$

$$\int (1-\cos^2(x))\sin(x)\cos^2(x)\,dx$$

$u = \cos(x)\,,\qquad du = -\sin(x)\,,\qquad -du = \sin(x)dx$

$$\int -(1-u^2)u^2 du$$

$$\int (-u^2 + u^4)du$$

$$-\frac{1}{3}u^3 + \frac{1}{5}u^5 + c$$

$$-\frac{1}{3}\cos^5(x) + \frac{1}{5}\cos^5(x) + c$$

11. By substitution $u = \sqrt{3x+1}$, find $\int \frac{x}{\sqrt{3x+1}}$

$du = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)dx = \frac{3dx}{2\sqrt{3x+1}},\qquad \frac{2}{3}du = \frac{dx}{\sqrt{3x+1}},\qquad x = \frac{u^2-1}{3}$

$$\int \frac{2(u^2-1)}{9}du$$

12. $\int \sin^2(x)$

$$\int \frac{1}{2}(1-\cos(2x))dx$$

$$\int \left(\frac{1}{2} - \frac{\cos(2x)}{2}\right)dx$$

$$\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$$

CHAPTER 22: INTEGRATION APPLICATION

Area Between Two Curves

When y = f(x)

$$\int_{x_1}^{x_2} [f(x) - g(x)]dx$$

- $f(x)$ is the graph with a greater value within the determined interval (top-most)
- $g(x)$ is the graph with a lesser value within the determined interval (bottom-most)

When x = f(y)

$$\int_{y_1}^{y_2} [f(y) - g(y)]dy$$

- $f(y)$ is the graph with a greater value within the determined interval (right-most)
- $g(y)$ is the graph with a lesser value within the determined interval (left-most)

Kinematics

- Direction
  - Above x-axis = positive
  - Below x-axis = negative
- Speed
  - Increasing
    - Velocity and Acceleration both positive or negative
  - Decreasing
    - Velocity and Acceleration have different signs

Equations

Position function: $s(t)$

Velocity function: $v(t) = \frac{ds}{dt}$

Acceleration function: $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Speed: $|v(t)|$

Displacement: $s(t_f) - s(t_i) = \int_{t_i}^{t_f} v(t)dt$

Distance: $s(t_f) + s(t_i) = \int_{t_i}^{t_f} |v(t)|dt$

The Disk Method

Revolved about the x-axis

$$\pi \int_{x_1}^{x_2} |f(x)|^2 dx$$

Things to do before you start:

- Isolate y
- Find x_1 and x_2
    - x_1 is almost always 0 unless you're dealing with $y = \sqrt{x - k}$, in which case it's k
    - x_2 is always k in $x = k$

Revolved about the y-axis

$$\pi \int_{y_1}^{y_2} |f(y)|^2 dy$$

- Isolate x
- Find y_1 and y_2
    - y_1 is almost always 0 unless you're dealing with $x = \sqrt{y - h}$, in which case it's h
    - y_2 is always h in $y = h$

Equations

Area: $A(x) = \pi r^2$

$dV = A(x)dx = \pi r^2 dx$

$r = f(x)$

$dV = A(x)dx = \pi f(x)^2 dx$

The Washer Method

Revolved about the x-axis

$$\pi \int_{x_1}^{x_2} [f^2(x) - g^2(x)]dx$$

- $f(x)$ is the graph with a greater value within the determined interval (top-most)
- $g(x)$ is the graph with a lesser value within the determined interval (bottom-most)
- x_1 is usually the first intersection between $f(x)$ and $g(x)$
- x_2 is usually the second intersection between $f(x)$ and $g(x)$

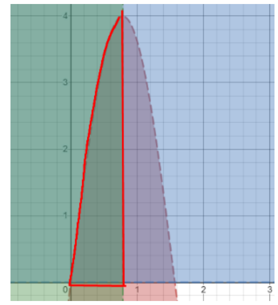
Revolved about the y-axis

$$\pi \int_{y_1}^{y_2} [f^2(y) - g^2(y)]dy$$

- $f(y)$ is the graph with a greater value within the determined interval (right-most)
- $g(y)$ is the graph with a lesser value within the determined interval (left-most)
- y_1 is usually the first intersection between $f(y)$ and $g(y)$
- y_2 is usually the second intersection between $f(y)$ and $g(y)$

Review Questions

1. Find the area of the region bounded by $y = 4\sin(2x)$, the x axis, and $x = \frac{\pi}{4}$



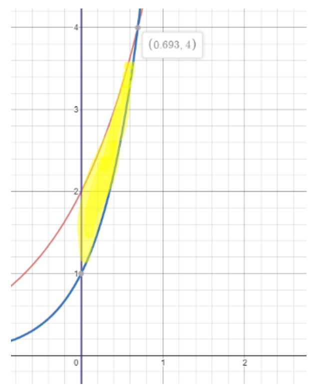
$$\int_0^{\frac{\pi}{4}} 4\sin(2x)\,dx$$

$$[-2\cos(2x)]\begin{matrix}\frac{\pi}{4}\\ 0\end{matrix}$$

$$\left[-2\cos\left(\frac{\pi}{2}\right)\right] - [-2\cos(0)]$$

2

2. Find the area of the region bounded by $y = 2e^x$, $y = e^{2x}$, and $x = 0$



$$2e^x = e^{2x}$$

$$2e^x - e^{2x} = 0$$

$$e^x(2 - e^x) = 0$$

$e^x = 0, \quad e^x = 2, \quad \therefore x = 0, \quad x = \ln(2)$

$$\int_0^{\ln(2)} (2e^x - e^{2x})dx$$

$$\left[2e^x - \frac{1}{2}e^{2x}\right]\begin{matrix}\ln(2)\\ 0\end{matrix}$$

$\frac{1}{2}$

3. A particle has velocity $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t. Determine if the speed is increasing or decreasing when t = 3. Show your reasoning.

$v(t) = t^3 - 10t^2 + 29t - 20,$ and $a(t) = 3t^2 - 20t + 29$

$$v(3) = 27 - 90 + 87 - 20 = 4$$

$$a(3) = 27 - 60 + 29 = -4$$

Decreasing

4. What is the displacement of the particle on the time interval [1, 5]

$$s(t) = \int t^3 - 10t^2 + 29t - 20\,dt = \frac{1}{4}t^4 - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t + c$$

$$\int_1^5 v(t)dt = s(5) - s(1)$$

$$\left[\frac{1}{4}5^4 - \frac{10}{3}5^3 + \frac{29}{2}5^2 - 20(5) + c\right] - \left[\frac{1}{4}1^4 - \frac{10}{3}1^3 + \frac{29}{2}1^2 - 20(1) + c\right]$$

$\frac{32}{3}$

5. A body is moving in a straight line. When it is s meters from a fixed point O on its line its velocity v is given by $v = -\frac{1}{s^2}$, where $s > 0$. Find the acceleration of the body when it is 50cm from O.

$$a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v$$

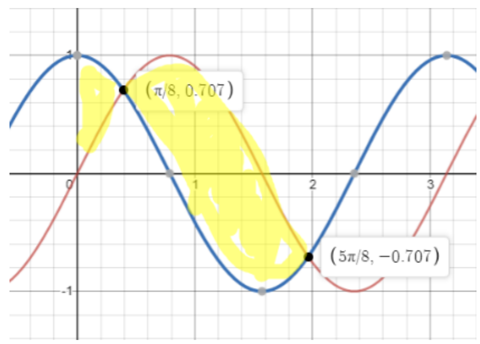
$$v = -\frac{1}{s^2}, \qquad \frac{dv}{ds} = 2s^{-3}$$

$$a = 2s^{-3} \times -\frac{1}{s^2} = \frac{-2}{s^5}$$

$$a(0.5) = -\frac{2}{(0.5)^5}$$

$$-64\frac{\text{m}}{s^2}$$

6. Given the region bounded by the x-axis, $f(x) = \cos(2x)$, and $g(x) = \sin(2x)$, find the first two intersections of x values in $[0, \pi]$. Give exact answers



$$\sin(2x) = \cos(2x)$$

$$\frac{\sin(2x)}{\cos(2x)} = 1$$

$$\tan(2x) = 1$$

$$2x = \tan^{-1}(1)$$

$$2x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{5\pi}{8}$$

7. Find the area between the two curves. Show your work in two integrals and give your answer in 3 significant figures.

$$\int_0^{\frac{\pi}{8}} (\cos(2x) - \sin(2x))dx + \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} (\sin(2x) - \cos(2x))dx$$

1.62 (use calculator)

8. Find the volume when the region bounded by the curve $y = x^2$ and $y = 4x$ is revolved about the y-axis

$$\sqrt{y} = x, \qquad \frac{y}{4} = x$$

$$\frac{y}{4} = \sqrt{y}$$

$$\frac{y^2}{16} = y$$

$$y = 16$$

$$V = \pi \int_0^{16} \left(\left(\sqrt{y}\right)^2 - \left(\frac{y}{4}\right)^2 \right) dy$$

$$V = \pi \int_0^{16} \left(\left(\sqrt{y}\right)^2 - \left(\frac{y}{4}\right)^2 \right) dy$$

$$\frac{128\pi}{3} = 134.041$$

9. The acceleration of a car is $\frac{1}{20}(80 - 2v)$, when its velocity is v. The car starts from rest. Find the equation of velocity in terms of time.

$$A(t) = \frac{dv}{dt} = \frac{1}{20}(80 - 2v)$$

$$dv = \frac{1}{20}(80 - 2v)dt$$

$$\frac{dv}{(80 - 2v)} = \frac{1}{20}dt$$

$$\int \frac{1}{(80 - 2v)}dv = \int \frac{1}{20}dt$$

$$-\frac{1}{2}\ln(80 - 2v) + c_1 = \frac{1}{20}t + c_2$$

When $t = 0, v = 0$ and $t = 0$

$$\therefore -\frac{1}{2}\ln\left(80 - 2(0)\right) + c = \frac{1}{20}(0), \qquad c = c_1 - c_2$$

$$\frac{1}{2}\ln(80) = c$$

$$\therefore -\frac{1}{2}\ln(80 - 2v) + \frac{1}{2}\ln(80) = \frac{1}{20}t$$

$$\ln(80) - \ln(80 - 2v) = \frac{1}{10}t$$

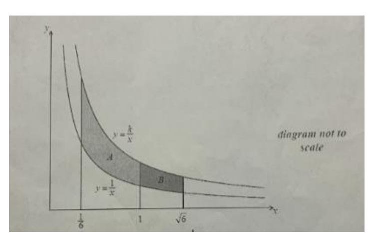
$$\ln(80 - 2v) = \ln(80) - \frac{1}{10}t$$

$$80 - 2v = e^{\ln(80) - \frac{1}{10}t}$$

$$-2v = -80 + 80e^{-\frac{1}{10}t}$$

$$v = 40 + 40e^{-\frac{1}{10}t}\,\frac{\text{m}}{\text{s}}$$

10. The graph of $y = \frac{1}{x}$ and $y = \frac{k}{x}$, where $k > 1$ is shown below. Find the area of A and B



$$A(A) = \int_{\frac{1}{6}}^{1} \left(\frac{k}{x} - \frac{1}{x}\right) dx$$

$$A(B) = \int_{1}^{\sqrt{6}} \left(\frac{k}{x} - \frac{1}{x}\right) dx$$

11. Find the ratio of the area of region A to the area of region B.

$$A(A) = [k\ln(x) - \ln(x)]\begin{matrix}1\\ \frac{1}{6}\end{matrix}$$

$[0] - \left[k \ln\left(\frac{1}{6}\right) - \ln\left(\frac{1}{6}\right)\right]$

$k \ln(6) - \ln(6)$

$\ln(6)\,(k-1)$

$A(B) = [k ln(x) - \ln(x)]\,\begin{matrix}\sqrt{6} \\ 1\end{matrix}$

$\left[k \ln(\sqrt{6}) - \ln(\sqrt{6})\right] - [0]$

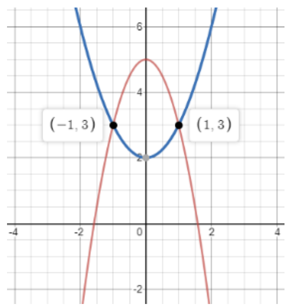
$\frac{k}{2}\ln(6) - \frac{1}{2}\ln(6)$

$\frac{1}{2}(\ln(6))(k-1)$

$ratio = \frac{A(A)}{A(B)} = \frac{\ln(6)\,(k-1)}{\frac{1}{2}(\ln(6))(k-1)}$

2

12. Write an integral expression for the volume of the solid revolved about the x-axis and enclosed by the functions $f(x) = 5 - 2x^2$ and $g(x) = x^2 + 2$



$5 - 2x^2 = x^2 + 2$

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x = -1$ and 1

$\pi \int_{-1}^{1} ((5-2x)^2 - (x^2+2)^2)dx$

CHAPTER 6: DIFFERENTIAL EQUATIONS AND APPLICATION

Integrals with Trig Identities (again)

Some trig identities that might come in handy

$\sin^2(x) + \cos^2(x) = 1$

$\cos(2x) = 1 - 2\sin^2(x)$

$\cos(2x) = 2\cos^2(x) - 1$

Integration by Parts

$$\int uv' = uv - \int vdu$$

Easier way to do this is with box method, where u and dv are given, then solve for du and v. The equation above can then be substituted with values found.

Tabular Method

Setting up a t-chart with u and dv on either side, then deriving on the left side (u) and integrating on the right side (dv) until the left side is zero or by "tabular substitution" (this isn't what it's actually called).

Tabular substitution

When the initial integration appears during tabular method, they can both be substituted for A and combined together, thus isolating A once more (examples are in the review questions)

Trigonometric Substitution

$$\sqrt{a^2 - x^2} = a\sin(\theta)$$

$$x^2 + a^2 = a\tan(\theta)$$

$$\sqrt{x^2 - a^2} = a\sec(\theta)$$

Things to do before you start:

- Find x
- Find dx
- The original $\sqrt{a^2 - x^2}$, $x^2 + a^2$, or $\sqrt{x^2 - a^2}$ by itself with a trig identity
- A suitable equation for θ

Separable Differential Equations

Isolate dy and dx on their respective sides

There are two ways to do this:

- Division/multiplication
- Distributive property, then divide/multiply

Finding a general solution

A general solution includes $+ c$. Your $+ c$ should usually be on the *opposite* side of the variable you are finding (finding y, $y = x + c$).

Finding a particular solution

A particular solution is when you're looking for the answer to A (which is usually e^c) or c. This can be solved by using the values provided in the question.

Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_s)$$

- T is the temperature at t minutes
- T_s is the surrounding temperature
- k is a constant

Modeling Situations with Differential Equations

$$\frac{dX}{dt} = kX$$

- $\frac{dX}{dt}$ is the rate of change of X with respect to t
- X is the manipulated variable
- k is a proportionality constant

Slope Fields

Slope fields are just a way to visualize the slope at a given point with a given equation. To find it, just plug in the x and y values. You'll probably have to do some work, as they're usually separable differential equations.

Review Questions

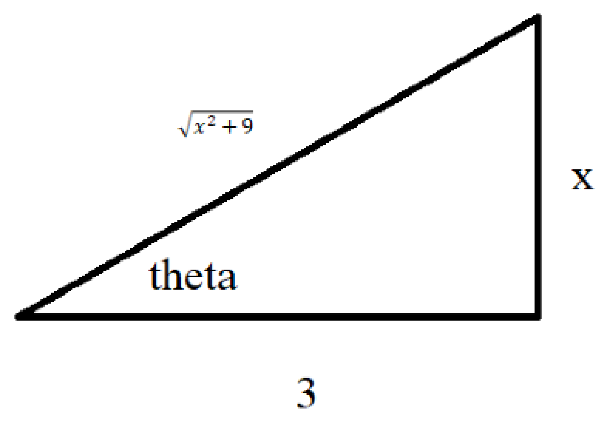
1. $\int \frac{1}{x^2+9} dx$

$x = 3\tan(\theta)$

$dx = 3\sec^2(\theta)\, d\theta$

$9\sec^2(\theta) = x^2 + 9$

$\theta = \operatorname{atan}\left(\frac{x}{3}\right)$



$$\int \frac{1}{9\sec^2(\theta)} 3\sec^2(\theta)\, d\theta$$

$$\int \frac{1}{3} d\theta = \frac{1}{3}\theta + c$$

$$\frac{1}{3}\operatorname{atan}\left(\frac{x}{3}\right) + c$$

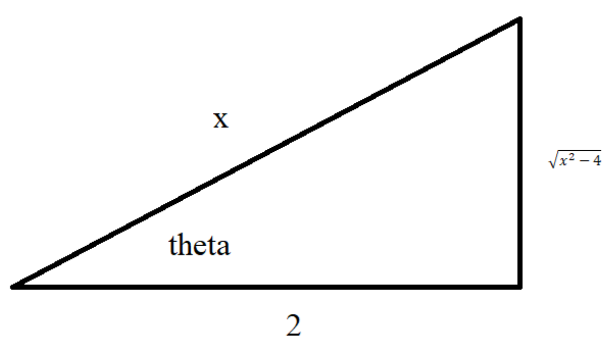
2. $\int \frac{\sqrt{x^2-4}}{x} dx$

$x = 2\sec(\theta)$

$dx = 2\sec(\theta)\tan(\theta)\, d\theta$

$\sqrt{x^2-4} = 2\tan(\theta)$

$\theta = \operatorname{asec}\left(\frac{x}{2}\right)$



$$\int \frac{2\tan(\theta)}{2\sec(\theta)} 2\sec(\theta)\tan(\theta)\, d\theta$$

$$\int 2\tan^2\theta\, d\theta$$

$$2\int (\sec^2\theta - 1)d\theta$$

$$2[\tan(\theta) - \theta] + c$$

$$2\left[\frac{\sqrt{x^2-4}}{2} - \operatorname{asec}\left(\frac{x}{2}\right)\right] + c$$

$\sqrt{x^2-4} - 2\,\text{asec}\left(\frac{x}{2}\right) + c$

3. $\int \frac{\sec^2(x)}{\sqrt{5\tan(x)+1}}\,dx$

$u = 5\tan(x) + 1, \qquad du = 5\sec^2(x)\,dx, \qquad \frac{1}{5}du = \sec^2(x)\,dx$

$$\frac{1}{5}\int u^{-\frac{1}{2}}du$$

$$\frac{1}{5}(2)\left(u^{\frac{1}{2}}\right) + c$$

$$\frac{2}{5}\sqrt{5\tan(x)+1} + c$$

4. $\int \frac{\ln(x)}{x^2}$

$u = \ln(x) \qquad dv = \frac{1}{x^2}$

$du = \frac{1}{x}dv \qquad v = -\frac{1}{x}$

$$-\frac{\ln(x)}{x} - \int -\frac{1}{x^2}dx$$

$$-\frac{\ln(x)}{x} + \int \frac{1}{x^2}dx$$

$$-\frac{\ln(x)}{x} - \frac{1}{x} + c$$

5. $\int x^3 e^{\frac{x}{4}}dx$

/	u	dv
$+$	x^3	$e^{\frac{x}{4}}$
$-$	$3x^2$	$4e^{\frac{x}{4}}$
$+$	$6x$	$16e^{\frac{x}{4}}$
$-$	6	$64e^{\frac{x}{4}}$
$+$	0	$256e^{\frac{x}{4}}$

$$4x^3e^{\frac{x}{4}} - 64x^2e^{\frac{x}{4}} + 384xe^{\frac{x}{4}} - 1536e^{\frac{x}{4}}$$

6. Find a general solution for $\frac{dy}{dx} = \frac{1}{y+x^2y}$.

$$(y + x^2y)dy = dx$$

$$y(1 + x^2)dy = dx$$

$$ydy = \frac{dx}{(1+x^2)}$$

$$\int ydy = \int \frac{dx}{(1+x^2)}$$

$$\frac{1}{2}y^2 = \text{atan}(x) + c$$

$$y = \sqrt{2\,\text{atan}(x) + c}$$

7. Find a general solution for $\frac{dy}{dx} = \frac{7x^3}{y^3}$.

$$y^3 dy = 7x^3 dx$$

$$\int y^3 dy = \int 7x^3 dx$$

$$\frac{1}{4}y^4 = \frac{7}{4}x^4 + c$$

$$y^4 = 7x^4 + 4c$$

$$y = \sqrt[4]{7x^4 + 4c}$$

8. Find a particular solution for $\frac{dy}{dx} = \frac{\sin(x)}{\cos(x)}$ when $y(0) = \frac{3\pi}{2}$.

$$dy = \tan(x)\,dx$$

$$y = -\ln(cos(x)) + c$$

$$y(0) = -\ln(cos(0)) + c = \frac{3\pi}{2}$$

$$\frac{3\pi}{2} = -\ln(1) + c$$

$$c = \frac{3\pi}{2}$$

$$y = -\ln(cos(x)) + \frac{3\pi}{2}$$

9. Given that $t^3 \times \frac{d\theta}{dt} = k - t$ and $\theta(2) = 0 = \theta(4)$, find k.

$$d\theta = \frac{(k-t)}{t^3}dt$$

$$\theta = \int \frac{(k-t)}{t^3}dt$$

$$\theta = \int (kt^{-3} - t^{-2})dt$$

$$\theta = -\frac{1}{2}kt^{-2} + t^{-1} + c$$

$$\theta(2) = 0 = -\frac{1}{2}k\left(\frac{1}{4}\right) + \frac{1}{2} + c$$

$$\frac{1}{8}k = \frac{1}{2} + c$$

$$k = 4 + 8c$$

$\theta(4) = 0 = -\frac{1}{2}k\left(\frac{1}{16}\right) + \frac{1}{4} + c$

$\frac{1}{32}k = \frac{1}{4} + c$

$k = 8 + 32c$

$4 + 8c = 8 + 32c$

$-4 = 24c$

$c = -\frac{1}{6}$

$k = 4 + 8\left(-\frac{1}{6}\right)$

$k = \frac{8}{3}$

10. Evaluate the proportionality constant k if the pressure P of a tire was 35 psi and is decreasing at 0.28 psi per minute at time $t = 0$. (HIGHLY DOUBT THIS IS ON THE FINAL)

$\frac{dP}{dt} = kP$

$\frac{dP}{dt} = kP$

$k = \frac{dP}{Pdt}$

$k = -\frac{0.28}{35}$

$k = -0.008$

11. Solve the differential equation with the conditions provided above.

$\frac{dP}{dt} = kP$

$\frac{dP}{P} = kdt$

$\int \frac{dP}{P} = \int kdt$

$\ln(P) = -0.008t + c$

$P = e^{-0.008t+c}$

$P = Ae^{-0.008t}$, when $A = e^c$

$35 = Ae^{-0.008(0)}$

$A = 35$

$P = 35e^{-0.008t}$

12. The acceleration of a body is given in terms of the displacement, s meters, as $a = 5s + 3$. Give a formula for the velocity as a function of the displacement given that when $s = 2$ meters, $v = 8$ meters per second.

$$a = 5s + 3 = \frac{dv}{ds}v$$

$$(5s + 3)ds = vdv$$

$$\int (5s + 3)ds = \int vdv$$

$$\frac{5}{2}s^2 + 3s + c = \frac{1}{2}v^2$$

$$\frac{5}{2}(2)^2 + 3(2) + c = \frac{1}{2}(8)^2$$

$$16 + c = 32$$

$$c = 16$$

$$\frac{5}{2}s^2 + 3s + 16 = \frac{1}{2}v^2$$

$$v^2 = 5s^2 + 6s + 32$$

$$v = \sqrt{5s^2 + 6s + 32}$$

CHAPTER 6+: MORE DIFFERENTIAL EQUATIONS

The Logistic Equation (NOT ON THE FINAL)

$$\frac{dy}{dt} = ky(1 - \frac{y}{L})$$

Partial Fractions (NOT ON THE FINAL)

This is taken from the Logistic Equation:

$$\frac{1}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$$

$$\frac{1}{y(L-y)} = \frac{\mathrm{A(L-y) + B(y)}}{y(L-y)}$$

$$1 = \mathrm{A(L-y) + B(y)}$$

Usually by this point you'll have numbers you can work with to isolate each variable.

Euler's Method

$$y_{n+1} = y_n + \Delta x(y'_n)$$

I visualize it with $y_n = y_{n-1} + \Delta x(y'_{n-1})$ – they're the same thing.

Reminder: Euler is always going to be less than the real integral, as the step size cannot be infinity, so its value will always be under the actual curve (think left-hand triangles in a concave-up curve)

Integrating Factors

$$\frac{dy}{dx} + py = Q$$

$$I(x)y = \int I(x)Q dx$$

$$I(x) = e^{\int P dx}$$

How to approach this problem:

- Set up the equation in the format $\frac{dy}{dx} + py = Q$
- Take p and find the integral of it, then place that as an exponent of e
- Multiply that number on all sides
- Then put that resultant in an equation $\frac{d}{dx}(\text{resultant} \times y)$
- Multiply dx to the other side and integrate that side
- Magic. (see review questions if you're still confused)

Homogenous Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$$y = vx, \qquad v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$F(v) = v + x\frac{dv}{dx}$$

Review Questions

1. Find the general solution of $xydx = (x - 5)dy$.

$$\frac{dy}{y} = \frac{x}{x-5}dx$$

$$\int \frac{dy}{y} = \int \frac{x}{x-5}dx$$

$$\ln(y) = \int \frac{x}{x-5}dx$$

$$u = x - 5, \qquad du = dx, \qquad x = u + 5$$

$$\ln(y) = \int \frac{u+5}{u}dx$$

$$\ln(y) = \int (1 + 5u^{-1})dx$$

$$\ln(y) = u + 5\ln(u) + c$$

$$y = e^{(x-5)+5\ln(x-5)+c}$$

$y = Ae^{(x-5)+5\ln(x-5)}$, when $A = e^c$

2. $y\frac{dy}{dx} = e^{x-3y}\cos(x)$

$$ydy = \frac{e^x}{e^{-3y}}\cos(x)\,dx$$

$$ye^{3y}dy = e^x\cos(x)\,dx$$

$$\int ye^{3y}dy = \int e^x\cos(x)\,dx$$

$$u = y, \qquad dv = e^{3y}$$

$$du = dy, \qquad v = \frac{1}{3}e^{3y}$$

$$\frac{1}{3}ye^{3y} - \int\frac{1}{3}e^{3y}dy = \frac{y}{3}e^{3y} - \frac{1}{9}e^{3y}$$

$$\frac{y}{3}e^{3y} - \frac{1}{9}e^{3y} = \int e^x\cos(x)\,dx$$

$$\begin{array}{lcc} / & u & dv \\ + & \cos(x) & e^x \\ - & -\sin(x) & e^x \\ + & -\cos(x) & e^x \end{array}$$

$$\int e^x\cos(x)\,dx = \cos(x)\,e^x + \sin(x)\,e^x - \int\cos(x)\,e^x dx$$

$$2\int e^x\cos(x)\,dx = \cos(x)\,e^x + \sin(x)\,e^x$$

$$\int e^x\cos(x)\,dx = \frac{1}{2}\cos(x)\,e^x + \frac{1}{2}\sin(x)\,e^x$$

$$\therefore \frac{y}{3}e^{3y} - \frac{1}{9}e^{3y} = \frac{1}{2}\cos(x)\,e^x + \frac{1}{2}\sin(x)\,e^x$$

Not quite sure where to go from here…

3. Find the particular solution of $y\ln(x) - x\frac{dy}{dx} = 0$ in terms of x when $y(e) = 12$.

$$y\ln(x) = x\frac{dy}{dx}$$

$$\frac{\ln(x)}{x}dx = \frac{dy}{y}$$

$$\int\frac{\ln(x)}{x}dx = \int\frac{dy}{y}$$

$$u = \ln(x)\,, \qquad du = \frac{1}{x}dx$$

$$\int udu = \ln(y)$$

$$\frac{1}{2}(\ln(x))^2 + c = \ln(y)$$

$$\frac{1}{2}(\ln(e))^2 + c = \ln(12)$$

$$c = \ln(12) - \frac{1}{2}$$

$$\frac{1}{2}(\ln(x))^2 + \ln(12) - \frac{1}{2} = \ln(y)$$

$$y = e^{\frac{1}{2}(\ln(x))^2 + \ln(12) - \frac{1}{2}}$$

$$y = 12e^{(\ln(x))^2 - \frac{1}{2}}$$

4. $\frac{dy}{dx} + 2y = e^{-x}$

$$Q = 2, \qquad I(x) = e^{\int 2dx} = e^{2x}$$

$$e^{2x}y' + e^{2x}2y = e^{2x}e^{-x}$$

$$\frac{d}{dx}(ye^{2x}) = e^{x}$$

$$d(ye^{2x}) = e^{x}dx$$

$$ye^{2x} = e^{x} + c$$

$$y = e^{-x} + \frac{c}{e^{2x}}$$

5. $xdy + ydx = \sin(x)\,dx$

$$x\frac{dy}{dx} + y = \sin(x)$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin(x)}{x}$$

$$Q = \frac{1}{x}, \qquad I(x) = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

$$x\frac{dy}{dx} + y = \sin(x)$$

$$\frac{d}{dx}(xy) = \sin(x)$$

$$d(xy) = \sin(x)\,dx$$

$$xy = \cos(x) + c$$

$$y = \frac{\cos(x) + c}{x}$$

6. $(x^2 + y^2)dx + xydy = 0$

$$(x^2 + y^2) + xy\frac{dy}{dx} = 0$$

$$x^2\left(1+\frac{y^2}{x^2}\right)+xy\frac{dy}{dx}=0$$

$$\left(1+\frac{y^2}{x^2}\right)+\left(\frac{y}{x}\times\frac{dy}{dx}\right)=0$$

$$\left(v\left(v+x\frac{dv}{dx}\right)\right)=-(1+v^2)$$

$$\left(v+x\frac{dv}{dx}\right)=-\frac{(1+v^2)}{v}$$

$$x\frac{dv}{dx}=\frac{-(1+v^2)-v^2}{v}$$

$$x\frac{dv}{dx}=\frac{-1-2v^2}{v}$$

$$\frac{1}{x}dx=\frac{-v}{1+2v^2}dv$$

$$\int\frac{1}{x}dx=\int\frac{-v}{1+2v^2}dv$$

$$u=1+2v^2,\qquad du=4vdv,\qquad -\frac{1}{4}du=-vdv$$

$$\ln(x)=-\frac{1}{4}\int\frac{du}{u}$$

$$\ln(x)=-\frac{1}{4}\ln(1+2v^2)+c$$

$$\ln(x)+\frac{1}{4}\ln(1+2v^2)=c$$

7. Find a general solution of $\frac{dy}{dx}=\frac{y^2-x^2}{6xy}$ using the substitution $\frac{y}{x}=v$.

$$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{6x^2v}$$

$$v+x\frac{dv}{dx}=\frac{x^2(v^2-1)}{6x^2v}$$

$$v+x\frac{dv}{dx}=\frac{(v^2-1)}{6v}$$

$$x\frac{dv}{dx}=\frac{(v^2-1)}{6v}-\frac{6v^2}{6v}$$

$$x\frac{dv}{dx}=\frac{-5v^2-1}{6v}$$

$$\frac{6v}{-5v^2-1}dv=\frac{dx}{x}$$

$$\int\frac{6v}{-5v^2-1}dv=\int\frac{dx}{x}$$

$u = -5v^2 - 1, \qquad du = -10v dv, \qquad -\frac{3}{5} du = 6v dx$

$-\frac{3}{5}\int \frac{du}{u} = \ln(x)$

$-\frac{3}{5}\ln(-5v^2 - 1) + c = \ln(x)$

$c = \ln(x) + \frac{3}{5}\ln(-5v^2 - 1)$

THAT'S IT GUYS – NO L'HOPITAL'S BC IT'S TAKING TOO LONG – HOPE THIS HELPED!