A COMPREHENSIVE

IB HL MATH 2 FIRST SEMESTER FINALS STUDY GUIDE

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Let me know in person if you have any questions!

CALCULUS REVIEW

Differentiation

Equation of Tangent/Normal Line

The derivative of a function gives the **slope** of its tangent at a point x. In order to find the equation of the tangent line, you need a couple things:

- The function itself
- A point (x, y) that lies on the function
  - Most scenarios will give you 'when x = …'
  - To solve this, find what the y value would be when plugging in $f(x)$
  - Differentiate the function
  - Plug in the x and y values
- A knowledge of differentiation techniques (chain rule, product rule, quotient rule)

Once done, the equation of the tangent line can be set up like this:

$$y = f'(x)(x - x_1) + y_1$$

Look familiar? It's just point-slope form: $y - y_1 = m(x - x_1)$

The normal line is a line that intersects perpendicularly with the tangent line of a function at a point x. To calculate the normal slope, just do $\frac{-1}{f'(x)}$, then follow the same steps for the full line equation.

Implicit Differentiation

Implicit differentiation usually occurs when differentiating an equation results in multiple appearances of $\frac{dy}{dx}$. To deal with this, **separate** the $\frac{dy}{dx}$ onto one side of the equation and **factor** it out, then dividing the distributed equation.

$$\frac{dy}{dx}equation = equation - \frac{dy}{dx}equation$$

$$\frac{dy}{dx}(equation + equation) = equation$$

$$\frac{dy}{dx} = \frac{equation}{equation + equation}$$

The goal of implicit differentiation is to get $\frac{dy}{dx}$ by itself.

Integration

Integration seeks to find the **area** under a function. It is essentially 'inverse differentiation', as one can go from $f'(x)$ to $f(x)$.

Standard Integrals

$$\int kdx = kx + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + c$$

$$\int \frac{1}{kx} dx = \frac{\log(kx)}{k} + c$$

$$\int \sin(kx)\, dx = -\frac{\cos(kx)}{k} + c$$

$$\int \cos(kx)\, dx = \frac{\sin(kx)}{k} + c$$

$$\int \sec^2(kx)\, dx = \frac{\tan(kx)}{k} + c$$

$$\int \sec(kx)\tan(kx)\, dx = \frac{\sec(kx)}{k} + c$$

Integration by u-Substitution

There are a couple critical steps to u-Substitution:

1. Determine what value will become u
2. Find the derivative of u, du.
3. du should almost always be a value present within the integration problem
4. Replace values within the integration problem with u and du.
5. The equation should be essentially fully replaced
6. Integrate with respect to standard integrals
7. Replace u with its corresponding value

$$\int \frac{value1}{value2}$$

$$u = value2, du = value1$$

$$\therefore \int \frac{du}{u}$$

$$\log(u) + c$$

$$\log(value2) + c$$

Integration by Parts

$$\int uv' = uv - \int vdu$$

Easier way to do this is with **box method**, where u and dv are given, then solve for du and v. The equation above can then be substituted with values found.

Integration by Trig Identities

Use the booklet for information regarding trig identities

If a sin-cosine integration has an odd power $\int \sin^3(x)\cos(x)\,dx$, then u-substitution is possible

Proving $\int \sec(x)\,dx$

$$\int \frac{\sec(x)}{1}\,dx$$

$$\int \frac{\sec(x)}{1} \times \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$\int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

$$u = \sec(x)+\tan(x)\,, \qquad du = \sec(x)\tan(x)+\sec^2(x)\,dx$$

$$\int \frac{1}{u}\,du = \ln(u)+c$$

$$\therefore \int \sec(x)\,dx = \ln(\sec(x)+\tan(x))+c$$

Integration by Trig Substitution

$$\sqrt{a^2-x^2} = a\sin(\theta)$$
$$x^2+a^2 = a\tan(\theta)$$
$$\sqrt{x^2-a^2} = a\sec(\theta)$$

Things to do before you start:

1. Find x
2. Find dx
3. Find original $\sqrt{a^2-x^2}$, x^2+a^2, or $\sqrt{x^2-a^2}$ by itself with a trig identity
4. Find a suitable equation for θ

Differential Equations

Separable Differential Equations (General Solution and Particular Solution)

Isolate dy and dx on their respective sides

There are two ways to do this:

- Division/multiplication
- Distributive property, then divide/multiply

Finding a general solution

A general solution includes $+\,c$. Your $+\,c$ should usually be on the *opposite* side of the variable you are finding (finding y, $y = x+c$).

Finding a particular solution

A particular solution is when you're looking for the answer to A (which is usually e^c) or c. This can be solved by using the values provided in the question.

Logistic Differential Equations and Newton's Law of Cooling

$$\frac{dy}{dt} = ky(1 - \frac{y}{L})$$

Linear Differential Equations

$$\frac{dy}{dx} + py = Q$$

$$I(x)y = \int I(x)Qdx$$

$$I(x) = e^{\int Pdx}$$

How to approach this problem:

1. Set up the equation in the format $\frac{dy}{dx} + py = Q$
2. Take p and find the integral of it, then place that as an exponent of e
3. Multiply that number on all sides
4. Then put that resultant in an equation $\frac{d}{dx}(\text{resultant} \times y)$
5. Multiply dx to the other side and integrate that side
6. Magic. (see review questions if you're still confused)

Homogeneous Differential Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$$y = vx, \qquad v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$F(v) = v + x\frac{dv}{dx}$$

Euler's Method

$$y_{n+1} = y_n + \Delta x({y'}_n)$$

I visualize it with $y_n = y_{n-1} + \Delta x({y'}_{n-1})$ as they're the same thing.

Slope Field

Slope fields are just a way to visualize the slope at a given point with a given equation. To find it, just plug in the x and y values. You'll probably have to do some work, as they're usually separable differential equations.

2nd Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

CHAPTER 15: VECTOR APPLICATION

Standards

A vector is a line that cuts through a defined dimension. The dimensions we work with in this class are 2D and 3D. A 2D vector can be visualized as a line on an XY plane. Think of a 2D vector like drawing a line on a piece of paper. It's restricted to the dimensions of the paper, which is 'two-dimensional'. A 3D vector can be visualized as a line on an XYZ plane. Think of a 3D vector like shooting an arrow, as it isn't restricted to two dimensions; it exists in the three-dimensional world.

Creating a Vector

If we have a point $A(a, b, c)$ and $B(d, e, f)$

$$\overrightarrow{AB} = (B - A) = \begin{pmatrix} d - a \\ e - b \\ f - c \end{pmatrix}$$

Component Form

2D

$$\vec{A} = \begin{pmatrix} a \\ b \end{pmatrix}$$

3D

$$\vec{A} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Notice how each dimension has its own slot in the vector component form.

Basis Form

$$\vec{A} = ai + bj + ck$$

Operations

Addition/Subtraction of Vectors

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a - d \\ b - e \\ c - f \end{pmatrix}$$

Multiplying/Dividing by a Number

$$2\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \times a \\ 2 \times b \\ 2 \times c \end{pmatrix}$$

Length of a Vector (aka Magnitude)

$$|\vec{A}| = \sqrt{(a)^2 + (b)^2 + (c)^2}$$

Unit Vector

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Dot Product

$$\vec{A} \cdot \vec{B}$$

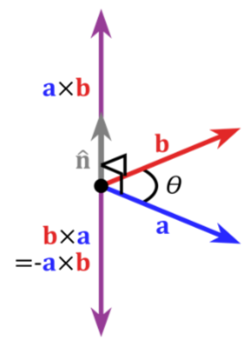
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a \times d \\ b \times e \\ c \times f \end{pmatrix}$$

Remember that when the dot product is equal to **zero**, the vectors are **perpendicular**

$$if\ \vec{A} \cdot \vec{B} = 0, \qquad \vec{A} \perp \vec{B}$$

Cross Product

Creates a vector that is **perpendicular** to both vectors provided.



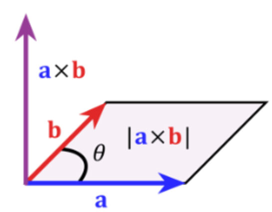
Use the box method to calculate cross product

$$\vec{A} = ai + bj + ck$$

$$\vec{B} = di + ej + fk$$

$$\begin{array}{ccc} i & j & k \\ - & - & - \\ a & b & c \\ d & e & f \end{array} = \begin{array}{l} i\big((b \times f) + (e \times c)\big) \\ -j\big((a \times f) + (d \times c)\big) \\ k\big((a \times e) + (d \times b)\big) \end{array}$$

Area



$\frac{1}{2}|\vec{u} \times \vec{v}| = \frac{1}{2}|\vec{u}||\vec{v}|\sin(\theta)$

Here is a good explanation if you don't understand

Lines/Planes in 3D

When a directional vector, $\vec{r}$, has form $\vec{r} = x_1 i + y_1 j + z_1 k$ and cuts through point (a, b, c), the following equations can be created.

Vectors

To determine a line you need:

- Two points
- A point and one directional vector

Cartesian Equation

$$\frac{x-a}{x_1} = \frac{y-b}{y_1} = \frac{z-c}{z_1}$$

Parametric Equation

$$x = a + x_1 t$$
$$y = b + y_1 t$$
$$z = c + z_1 t$$

Vector Equation

$$\vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Planes

To determine a plane you need:

- Three non-colinear points
- One point and two nonparallel directional vectors
- One point and a normal vector to the plane

Cartesian Equation

$$Ax + By + Cz = D$$

If you're given:

- A normal vector
- A point

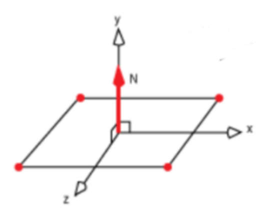
Then switch the vector to the cartesian plane equation and **plug** in the point coordinates, giving you the D value.

$$Ax + By + Cz = Ax_1 + By_1 + Cz_1 = D$$

Vector Equation

$$\vec{r} = \vec{a} + \lambda\vec{b} + t\vec{c}$$
$$n \cdot \overrightarrow{AR} = 0$$
$$r \cdot n = a \cdot n$$

Normal Vector



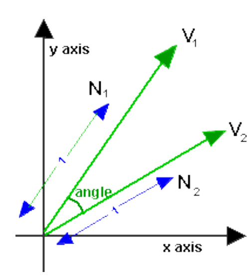
$$\vec{n} = Ai + Bj + Ck$$

The normal vector has the same variables as the cartesian equation.

Angles in between Lines/Planes

Line and Line

The angle that is calculated indicates the angle between the direction that both vectors point

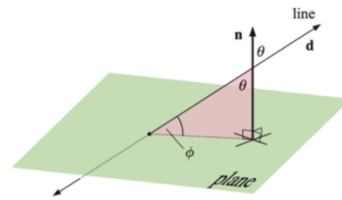


$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

This is a simplified way to calculate the angle:

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Line and Plane



Imagine a directional vector cutting through a piece of paper. That angle, ϕ, can be seen on the visual on the right.

Notice how $\vec{n}$, the normal vector, points up while $\vec{d}$, the directional vector, cuts up and to the left. The angle formed becomes θ.

The red triangle formed requires 180°. Since it's a right triangle, that means $\theta + \phi = 90°$

There are two ways you can do this:

1. The standard method of finding the angle between two vectors

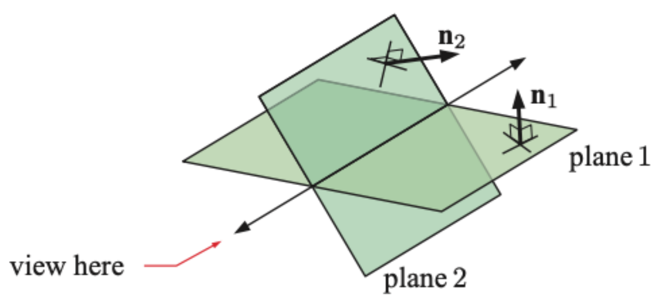
$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

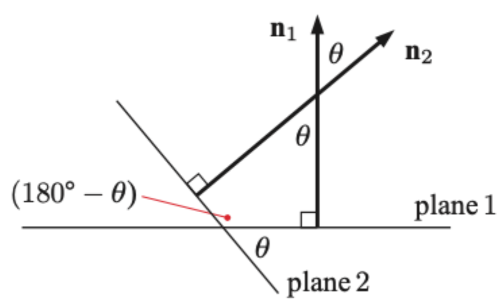
   a. Once you do that, however, you need to subtract by 90. This is because we found θ, not ϕ.

2. The 'shortcut' method
   a. Replace cosine with sine, and solve

$$\sin(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Plane and Plane





Steps:

1. Take the normal vectors from both planes
2. Find θ, the angle between both vectors
   a. This is done with the standard method

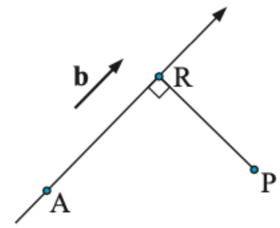
$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

3. Subtract θ by 180°

Shortest Distance between Lines/Planes/Points

In order to find the shortest distance, a perpendicular vector must be created. The point where the perpendicular intersects with the original vector is called the foot.

Line and Point



You will need:

- The parametric equation of a vector $\vec{r}$
- The coordinates of the point P

Steps:

1. Gather what's needed first
2. Create a vector PF from the point to the foot
    a. The component form should contain a variable
    b. This variable indicates a point on $\vec{r}$
    c. Think of it as $y = x + 5$, where $x = 5$
3. Set the dot product between $\overrightarrow{PF}$ and $\vec{r}$ to be zero
    a. This means that the point selected on $\vec{r}$ connects with P, making $\overrightarrow{PF}$ a perpendicular vector to $\vec{r}$
4. Plug t back into the parametric equations for the vector $\vec{r}$
5. This will give you the coordinates of the foot F, or the point where $\vec{r}$ and $\overrightarrow{PF}$ are perpendicular
6. Now, you have the coordinates of P and the coordinates of F
7. We know the 2D distance formula to be $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Therefore, the 3D distance formula will include the z axis

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Plane and Line

You will need:

- A plane P and a line $\vec{L}$ that is parallel to the plane but not on the plane
    - You will most likely get the equation for the line in cartesian or polar
        - Acquire the point that lies on the plane as well
- You will know if they are parallel if the normal vector, $\vec{n}$, of the plane is perpendicular to the vector

$$\vec{L} \cdot \vec{n} = 0$$

Steps:

1. Create a new plane in which $\vec{L}$ lies on.
    a. This means using the Cartesian equation for P but without the D value
2. Plug in the point value (that lies on the plane) into the Cartesian equation for your new plane, P_2.
3. You now have two planes!
    a. P, given

    b. P_2, where $\vec{L}$ and the point that lies on $\vec{L}$ lie
4. Refer to **Plane and Plane** to solve the distance.

Plane and Plane

You will need:

- Two parallel planes
    - Prove this by comparing their normal vectors. If their normal vectors are parallel, then the planes are parallel
- The Cartesian equations of both planes
    - This must include the D value

Steps:

1. Understand the distance equation for planes
$$\vec{r} \cdot \vec{n} = D$$
    a. The D value is the same value as the one in the cartesian equation
    b. $\vec{n}$ is the normal vector in basis form
    c. Leave $\vec{n}$ as it is
2. Convert $\vec{n}$ to a unit vector. This means dividing both sides by its magnitude
$$\vec{r} \cdot \frac{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}{\sqrt{a^2 + b^2 + c^2}} = \frac{D}{\sqrt{a^2 + b^2 + c^2}}$$
    a. Apply this to the D values on both planes
3. Understand the continued distance equation
$$Distance = the\ difference\ between\ D_1 and\ D_2$$
$$Distance = |D_1 - D_2|$$

Intersections of Lines/Skewed Lines

Distinctions between Intersecting/Skew/Parallel Lines

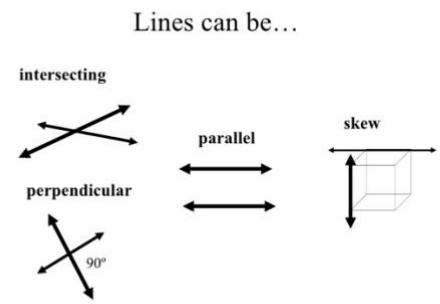
Here's an easy way to visualize the differences between intersecting, skew, and parallel lines.

- Intersecting and Parallel Lines are on the same plane.
    - Think of it as a piece of paper.
    - You can draw two parallel lines on a piece of paper – they will never intersect
    - You can also draw intersecting lines on a piece of paper – an X is a prime example of such.
- Skew lines do not exist on the same plane.
    - Back to the paper. Think of one line cutting perpendicularly through the paper, while one point is drawn on the paper (it lies on the plane)

- They may share intersections on two out of three dimensions (so if you view it from one angle it looks like they're intersecting) but in reality they will never touch
- Another way to visualize it is the faces of a cube. If you draw a line on one side of a cube and draw another one another side, they will never touch and are on different planes.

Summarized Definitions:

- Intersecting: on the same plane and meets at one point
- Parallel: on the same plane and will never meet
- Skew: on completely different planes and will never meet



How to Classify Lines with Algebra

You will need:

- Two vectors l_1 and l_2

Steps:

1. Put both l_1 and l_2 into parametric form
    a. You will need to find the coordinates given for each vector l_1 and l_2
    b. You should have something along the lines of:

$$l_1 \rightarrow \begin{array}{c} x = a + x_1 t \\ y = b + y_1 t, \\ z = c + z_1 t \end{array} \qquad l_2 \rightarrow \begin{array}{c} x = d + x_2 \lambda \\ y = e + y_2 \lambda \\ z = f + z_2 \lambda \end{array}$$

2. Combine corresponding variables from each side

$$x = a + x_1 t = d + x_2 \lambda$$
$$y = a + y_1 t = d + y_2 \lambda$$
$$z = a + z_1 t = d + z_2 \lambda$$

3. Move variables to one side and the numbers to the other
4. Choose two equalities and use system of equations to find one variable
    a. Find the other variable as well
5. Now that you have the values of the two variables, plug them back into the original parametric equations from step 1
6. Compare the values from each line
7. Results
    a. Coincident: basis forms and coordinates are multiples of each other

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = k \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \; and \; \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = k \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

    b. Parallel: basis forms are multiples of each other, but coordinates are not

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = k \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

c. Skew: two values are equal but one is not
d. Intersecting: all three values are equal

Intersection of Planes

Refer to **Unique/Infinity/No Solutions for Three Planes**.

Row Operations

Augmented Matrix

$$\begin{bmatrix} a & c & | & e \\ b & d & | & f \end{bmatrix}$$

Row Reduction

A method for solving systems of linear equations. This method uses augmented matrices.

How to use Row Reduction

Row reduction attempts to simplify a specific row by comparing that row with another row in the matrix. Think of it like doing system of equations without a matrix.

$$4x + 4y = 8$$
$$2x + 5y = 10$$

Multiply the $2x + 5y = 10$ by -2, then compare again

$$-4x - 10y = -20$$
$$4x + 4y = 8$$

Then, add the two equations together

$$(-4 + 4)x + (-10 + 4)y = -20 + 8$$
$$0x - 6y = -12$$
$$-6y = -12$$
$$y = 2$$

Now let's visualize this in matrix form:

$$\begin{bmatrix} 4 & 4 & | & 8 \\ 2 & 5 & | & 10 \end{bmatrix}$$

We're going to combine $-2 \times row2$ with row 1. If you code, you may see this as $row2 = (R2 \times -2) + R1$.

$$\begin{bmatrix} 4 & 4 & | & 8 \\ -4 & -10 & | & -20 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 & | & 8 \\ 0 & -6 & | & -12 \end{bmatrix}$$

Now we can simplify row 2 by dividing by -6

$$\begin{bmatrix} 4 & 4 & | & 8 \\ 0 & 1 & | & 2 \end{bmatrix}$$

The second column represents the y value, therefore $1y = 2$.

2D

1. Get the lower left corner, b, to zero.

$$\begin{bmatrix} a & c & | & e \\ b & d & | & f \end{bmatrix}$$

$$\begin{bmatrix} a & c & | & e \\ 0 & d & | & f \end{bmatrix}$$

2. Simplify d to 1.

$$\begin{bmatrix} a & c & | & e \\ 0 & 1 & | & f \end{bmatrix}$$

3. Interact R1 and R2 to get c to 0.
    a. The operation should look something like:

$$R1 = (number)R1 - (number)R2$$

$$\begin{bmatrix} a & 0 & | & e \\ 0 & 1 & | & f \end{bmatrix}$$

4. Simplify a to 1.

$$\begin{bmatrix} 1 & 0 & | & e \\ 0 & 1 & | & f \end{bmatrix}$$

3D

1. Get the bottom left corner, c, to 0.
    a. This is usually by interacting R1 and R3.

$$\begin{bmatrix} a & d & g & | & j \\ b & e & h & | & k \\ c & f & i & | & l \end{bmatrix}$$

$$\begin{bmatrix} a & d & g & | & j \\ b & e & h & | & k \\ 0 & f & i & | & l \end{bmatrix}$$

2. Get the middle left, b, to 0.
    a. This is usually by interacting R1 and R2.

$$\begin{bmatrix} a & d & g & | & j \\ b & e & h & | & k \\ 0 & f & i & | & l \end{bmatrix}$$

$$\begin{bmatrix} a & d & g & | & j \\ 0 & e & h & | & k \\ 0 & f & i & | & l \end{bmatrix}$$

3. Get the middle bottom, f, to 0.
    a. This is usually by interacting R2 and R3.

$$\begin{bmatrix} a & d & g & | & j \\ 0 & e & h & | & k \\ 0 & 0 & i & | & l \end{bmatrix}$$

4. Simplify i to 1

$$\begin{bmatrix} a & d & g & | & j \\ 0 & e & h & | & k \\ 0 & 0 & 1 & | & l \end{bmatrix}$$

5. Interact R2 and R3 to get h to 0.
   a. The operation should look something like:

$$R2 = (number)R2 - (number)R3$$

$$\begin{bmatrix} a & d & g & | & j \\ 0 & e & 0 & | & k \\ 0 & 0 & 1 & | & l \end{bmatrix}$$

6. Simplify e to 1.

$$\begin{bmatrix} a & d & g & | & j \\ 0 & 1 & 0 & | & k \\ 0 & 0 & 1 & | & l \end{bmatrix}$$

7. Interact R1 and R3 to get g to 0.
   a. The operation should look something like:

$$R1 = (number)R1 - (number)R3$$

$$\begin{bmatrix} a & d & 0 & | & j \\ 0 & 1 & 0 & | & k \\ 0 & 0 & 1 & | & l \end{bmatrix}$$

8. Interact R1 and R2 to get d to 0.
   a. The operation should look something like:

$$R1 = (number)R1 - (number)R2$$

$$\begin{bmatrix} a & 0 & 0 & | & j \\ 0 & 1 & 0 & | & k \\ 0 & 0 & 1 & | & l \end{bmatrix}$$

9. Simplify a to 1.

$$\begin{bmatrix} 1 & 0 & 0 & | & j \\ 0 & 1 & 0 & | & k \\ 0 & 0 & 1 & | & l \end{bmatrix}$$

Unique/Infinity/No Solutions for Three Planes

Standards

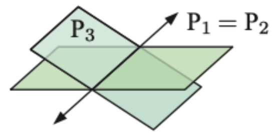
When a matrix looks like this:

$$\begin{bmatrix} 1 & d & g & | & j \\ 0 & 1 & h & | & k \\ 0 & 0 & i & | & l \end{bmatrix}$$

intersect. See the visual below: Then there are three possible scenarios:

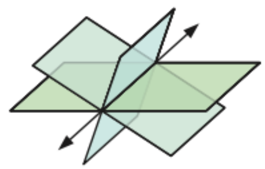
1. Infinite Solution: $i = 0,\ l = 0$
   a. This means that two of the planes are the same (coincident), and therefore the resultant two planes will make a line when they intersect.

(2) two coincident and one intersecting



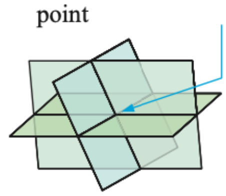
b. It could also mean that the planes all just make a line when they intersect

(7) all meet in a common line



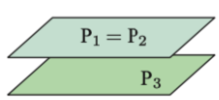
2. Unique Solution: $i = 1,\ l = some\ number$
   a. This means that Plane 3 is unique and the three planes intersect/meet at a specific point in space. See the visual below:

(6) all meet at the one point

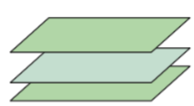


3. No Solution: $i = 0,\ l = some\ number$
   c. This means that there are multiple or no areas where the three planes intersect, Since $0 \times i = l$ will never give a real value for either i nor l.

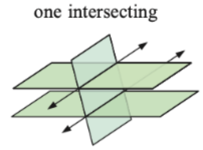
(3) two coincident and one parallel



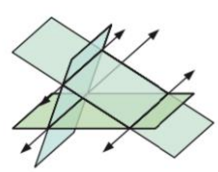
(5) all three parallel



(4) two parallel and one intersecting



(8) the line of intersection of any two is parallel to the third plane.



You can see more examples of triple plane interaction arrangements in the packet titled "2x2 Row Reduction" that was given on 12/9, or on page 467 in the IB Math HL textbook.

CHAPTER 16: COMPLEX NUMBERS

Standards

Understanding Polar Coordinates

Complex points contain a **real number** and an **imaginary number**. They lie on the **Argand Plane**. If this confuses you, refer to the following comparisons in the x-y plane:

Normal	Complex
XY Plane	Argand Plane
X Axis	Real Axis
Y Axis	Imaginary Axis
X-axis reflection	Conjugate
Hypotenuse	Modulus
How much the hypotenuse has rotated	Argument

Argument/Modulus

Refer to **Polar/Cartesian/Euler Forms of Complex Numbers**

Polar/Cartesian/Euler Forms of Complex Numbers

Cartesian Form

- Complex Number: $z = a + bi$
    - a represents the displacement of the point on the **real axis**
    - b represents the displacement of the point on the **imaginary axis**
- The Conjugate of z: $z^* = a - bi$
    - Think about it as reflecting across the x axis
- Modulus: $|z| = \sqrt{a^2 + b^2}$
    - Think about it as the hypotenuse

Polar Form

- Complex Number: $z = r\cos(\theta) + ir\sin(\theta) = rcis(\theta)$
- Conjugate of z: $z^* = r\cos(-\theta) + ir\sin(-\theta) = rcis(-\theta)$
- Modulus: $r = |z| = \sqrt{a^2 + b^2}$
- Argument: $Arg(z) = \theta,\ where -\pi \leq \theta \leq \pi$
    - Argument describes the amount (in radians) the hypotenuse has rotated counterclockwise.

Euler Form

- Complex Number: $rcis(\theta) = re^{i\theta}$

- If your e doesn't have an i, **do not** convert it into cis form

Properties of $z = rcis(\theta)$

$$cis(\theta) \times cis(\beta) = cis(\theta + \beta)$$
$$\frac{cis(\theta)}{cis(\beta)} = cis(\theta - \beta)$$
$$cis(\theta + 2\pi k) = cis(\theta), \qquad where\ k \in \mathbb{Z}$$

De Moivre's Theorem

$$[r(\cos(\theta) + i\sin(\theta))] = r^n(\cos(n\theta) + i\sin(n\theta))$$
$$[rcis(\theta)] = r^n\big(cis(n\theta)\big)$$

You will not need to prove De Moivre's theorem by math induction on the final

Roots of Complex Numbers

$$(a + bi)^{\frac{1}{n}} = [r(\cos(\theta) + i\sin(\theta))]^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$
$$where\ k = 0, 1, 2, \dots, n - 1$$

If you're given an equation $z^n - a = 0$:

1. Move a to the other side
$$z^n = a$$
2. Root the equality by n
$$z = a^{\frac{1}{n}}$$
3. Convert a to $acis(0)$
   a. This is because $cis(0) = 1$
   b. $a = a \times 1 = a \times cis(0)$
$$z = \big(acis(0 + 2\pi k)\big)^{\frac{1}{n}}$$
4. Continue with De Moivre's Theorem

MATH INDUCTION

Standards

You will need:

- An equation $P(n)$ or a conjecture to prove

Steps to perform induction (with correct notation):

1. Show that the statement is true for an initial case, $n = 1$
2. Assume that the statement is true for $n = k$ where $k \in Z^+$
3. Prove that the statement is true for $n = k + 1$
4. $\therefore$ the statement is true for $n \in Z^+$

Series

1. **When $\boldsymbol{n = 1}$…**
    a. Solve the left hand side for when $n = 1$
    b. Solve the right hand side for when $n = 1$
    c. Compare
    d. **$\therefore$ $\boldsymbol{P(n)}$ is true for $\boldsymbol{n = 1}$**
2. **When $\boldsymbol{n = k}$ where $\boldsymbol{k = Z^+}$…**
    a. Replace the n values with k
    b. **Assume [insert $\boldsymbol{P(k)}$ fully written out] is true**
3. **If $\boldsymbol{n = k + 1}$…**
    a. Write down what you're looking for
        i. This is usually the right hand side
        ii. Replace the n with $k + 1$
    b. **$\boldsymbol{P(n+1) = P(n) + LHS(k+1)}$**
        i. Assuming that $P(n)$ is a series, the next value will always contain what's stated in the LHS
            1. Example: when we did $\sum_{i=1}^{k} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{k+1} i$ can be split into $\sum_{i=1}^{k} i + (k + 1)$
    c. **Simplify the expression in the effort to reach what you wrote down at the beginning of step 3**
4. **$\therefore$ $\boldsymbol{P(n)}$: [write out whole statement] is true for all [limitations]**

Trigonometry

Trigonometry Induction is nearly identical to Series Induction. The only difference is that you may need to apply the following identity for difference as a product:

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Try to think of this identity with α and β rather than x and y, since you're usually given x variables in your problem, which are **not** the same as α and β.

The main difference of Trigonometric Induction is the third step.

1. Same
2. Same
3. **If $\boldsymbol{n = k + 1}$...**
    a. **Try your best to place the equation into the form containing $\boldsymbol{2\cos(value1)\sin(value2)}$.**
    b. Then, understand the following relations
        i. $value1 = \frac{\alpha+\beta}{2}$
        ii. $value2 = \frac{\alpha-\beta}{2}$
    c. Use these relations to solve for α and β
    d. Simplify
4. Same

Divisibility

Step 2 and Step 3 are different in Divisibility induction

1. Same
2. **When $\boldsymbol{n = k}$ where $\boldsymbol{k = Z^+}$...**
    a. Take the divisibility number (example: divisible by 3) and create a variable A that contains the divisibility number (example: $3A$)
        i. This means that when you want to find A by itself, you divide $P(n)$ by 3 and thus A is possible to be found.
        ii. This means that $P(n)$ is divisible by 3!
    b. **Assume that $\boldsymbol{P(n)}$ is divisible by 3**
    c. **[write $\boldsymbol{P(n)}$ out] = $\boldsymbol{nA}$**
        i. n is the divisibility amount that was mentioned before
3. **If $\boldsymbol{n = k + 1}$...**
    a. Replace the n with $k + 1$
    b. Simplify through substitution (examples included)
        i. $k^3 + 2k = 3A$
            1. $(k + 1)^3 = 2(k + 1)$
            2. $k^3 + 3k^2 + 3k + 1 + 2k + 2$
            3. $k^3 + 2k + 3k^2 + 3k + 3$
                a. Remember that $k^3 + 2k = 3A$
            4. $3A + 3k^2 + 3k + 3$
                a. Simplify
            5. $3(A + k^2 + 3k + 3)$

ii. $5^k - 1 = 4A$
1. $5^{k+1} - 1$
2. $5 \cdot 5^k - 1$
   a. Remember that $5^k - 1 = 4A$, therefore $5^k = 4A + 1$
3. $5(4A + 1) - 1$
4. $20A + 5 - 1$
5. $20A + 4$
6. $4(5A + 1)$

4. Same

Calculus

The Calculus Induction problems we have been given involve differentiation (taking the derivative).

This means that the function is usually $f^n(x) = something$.
This is **not** an exponent. It is the amount of times you need to take the derivative.

The only different step is the third one.

1. Same
2. Same
3. **If $\boldsymbol{n = k + 1}$…**
   a. When you see $f^{k+1}(x)$, **do not** change the variable on the RHS to $k + 1$.
   b. This is because you are taking the **derivative**.
      i. You should still write down what you're looking for, which **does** contain a replacement of the RHS to $k + 1$.
   c. Therefore, the next step would be to take the derivative
   d. Simplify and compare
4. Same

Complex Numbers

Refer to the packet titled "Complex Numbers" that was given on 12/6.

Inequality

NO (its not on the final lol)