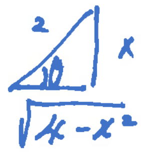
Integration Techniques (IB Questions Solution)

#1.

(a) $\int \sqrt{4-x^2}\,dx$



$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$= \int 4\cos^2\theta\, d\theta$

$= 2\int (1+\cos 2\theta)\, d\theta$

$= 2\left[\theta + \sin\theta\cos\theta\right] + C$

$\int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx \Rightarrow 2\left[\arcsin\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right]_{x=0}^{\sqrt{2}}$

$= 2\left[\arcsin\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{4-2}}{2}\right) - \arcsin 0 - \left(\frac{0}{2}\right)\left(\frac{\sqrt{4}}{2}\right)\right]$

$= 2\left[\frac{\pi}{4} + 1\right] = \boxed{\frac{\pi}{2} + 2}$

(b) $\int \arcsin x\, dx = x \cdot \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$

$\oplus \qquad \ominus\int$

$\Rightarrow x\arcsin x + \frac{1}{2}\int u^{-\frac{1}{2}}\,du \qquad u = 1-x^2$

$du = -2x\,dx \qquad -\frac{1}{2}du = x\,dx$

$= \boxed{x\arcsin x + \sqrt{1-x^2}} = f(x)$

$\int_0^{0.5} \arcsin x\, dx = \left[x\arcsin x + \sqrt{1-x^2}\right]_{x=0}^{x=0.5}$

$= \frac{1}{2}\arcsin\left(\frac{1}{2}\right) + \sqrt{\frac{3}{4}} - 0 - \sqrt{1}$

$= \frac{1}{2}\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

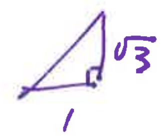
(c) $t = \tan\theta \qquad dt = \sec^2\theta\, d\theta.$

$$\int \frac{d\theta}{3\cos^2\theta + \sin^2\theta} \begin{matrix}\div \cos^2\theta \\ \div \cos^2\theta\end{matrix} \Rightarrow \int \frac{\sec^2\theta\, d\theta}{3 + \tan^2\theta}.$$

$$= \int \frac{dt}{t^2+3} = \frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + c$$

$$\theta = \frac{\pi}{4} \qquad t = \tan\left(\frac{\pi}{4}\right) = 1$$

$$\theta = 0 \qquad t = \tan(0) = 0$$



$$\Rightarrow \int_0^1 \frac{dt}{t^2+3} = \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right)\right]_{t=0}^{t=3}$$

$$= \frac{1}{\sqrt{3}}\left[\arctan(\sqrt{3}) - \arctan(0)\right] = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3}\right] = \frac{\pi}{3\sqrt{3}}$$

#2. $\int_1^{\sqrt{3}} \sqrt{4-x^2}\, dx.$ (Same as #1 (a))

$$= 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\, \frac{\sqrt{4-x^2}}{2}\right]_{x=1}^{x=\sqrt{3}}$$

$$= 2\left[\arcsin\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\, \frac{\sqrt{4-3}}{2} - \arcsin\left(\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right]$$

$$= 2\left[\frac{\pi}{3} + \cancel{\frac{\sqrt{3}}{4}} - \frac{\pi}{6} - \cancel{\frac{\sqrt{3}}{4}}\right] = 2\cdot\frac{\pi}{6} = \boxed{\frac{\pi}{3}}$$

#3. $\displaystyle\int \frac{(u-2)^3}{u^2}\,du = \int \frac{u^3-6u^2+12u-8}{u^2}\,du$

1
1 1
1 2 1
1 3 3 1

$u^3+(3)(u^2)(-2)+(3)(u)(-2)^2$
$+(-2)^3$

$u = x+2$

$x = u-2$

$dx = du$

$= \int (u - 6 + 12u^{-1} - 8u^{-2})\,du$

$= \boxed{\frac{1}{2}(x+2)^2 - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + C}$

#4. (a) $\int (1+\tan^2 x)\,dx = \int \sec^2 x\,dx = \boxed{\tan x + C}$

(b) $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

$= \boxed{\frac{1}{2}x - \frac{1}{2}\sin x\cos x + C}$

#5. $\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

$\left(\begin{array}{ll} u = \ln x & dv = x^3 \\ du = \frac{1}{x}dx & \oplus\; v = \frac{1}{4}x^4 \\ & \ominus \end{array} \right)$

$= \frac{1}{4}x^4 \ln x - \frac{x^4}{16} + C$

$\displaystyle\int_1^2 x^3 \ln x\,dx = \left[\frac{1}{4}x^4 \ln x - \frac{x^4}{16}\right]_{x=1}^{x=2}$

$= \left[4\ln 2 - 1\right] - \left[0 - \frac{1}{16}\right]$

$= \boxed{4\ln 2 - \frac{15}{16}}$