#1. $\lim_{x\to 0} \frac{\ln(1+\sin x) - x}{x^2} \quad \left(\frac{0}{0}\right)$

l'Hopital's Rule

$$= \lim_{x\to 0} \frac{\left(\frac{1}{1+\sin x}\right)(\cos x) - 1}{2x} \quad \left(\frac{0}{0}\right)$$

l'Hopital's Rule again

$$= \lim_{x\to 0} \frac{(-\sin x)(1+\sin x) - (\cos x)(\cos x)}{2}$$

$$= \lim_{x\to 0} \frac{-\sin x - \sin^2 x - \cos^2 x}{2} = \lim_{x\to 0} \frac{-\sin x - (\sin^2 x + \cos^2 x)}{2}$$

$$= \lim_{x\to 0} \frac{-\sin x - 1}{2} = \boxed{-\frac{1}{2}}$$

#2 b) $\lim_{x\to\infty} \int_0^{\infty} \frac{x^2}{e^{x}} \quad \left(\frac{\infty}{\infty}\right)$

u	dv
x^2	$+\ e^{-x}$
$2x$	$-\ -e^{-x}$
2	$+\ e^{-x}$
0	$-e^{-x}$

$$= \lim_{a\to\infty}\left[\frac{-a^2}{e^a} - \frac{2a}{e^a} - \frac{2}{e^a}\right] - \left[-\frac{2}{1}\right]$$

$$= \lim_{a\to\infty}\left[\frac{-2}{e^a} - \frac{2}{e^a} - \frac{2}{e^a}\right] - [-2] = \boxed{2}$$

0 0 0

Converges to '2'.

a) $\lim_{x\to\infty} \frac{x^2}{e^x} \quad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}$$

#3

a) $f'(x) = \frac{e^x - e^{-x}}{2}$ $\qquad g'(x) = \frac{e^x + e^{-x}}{2}$

b) Skip

②

#3 (c) $\lim_{x\to 0} \frac{1-\left[\frac{e^x+e^{-x}}{2}\right]}{x^2}$ $\left(\frac{0}{0}\right)$

l'Hopital

$= \lim_{x\to 0} \frac{-\left[\frac{e^x-e^{-x}}{2}\right]}{2x}$ $\left(\frac{0}{0}\right)$ $= \lim_{x\to 0} \frac{\left[e^{-x}-e^{x}\right]}{4x}$ $\left(\frac{0}{0}\right)$

l'Hopital

$= \lim_{x\to 0} \frac{-e^{-x}-e^{x}}{4} = \frac{-2}{4} = \boxed{\frac{-1}{2}}$ $\qquad u = \frac{e^x+e^{-x}}{2} = 1$

(d) $f(x) = u \quad du = g(x)\cdot dx \qquad x=0 \Rightarrow u=1$

$x \to \infty \Rightarrow u \to \infty$

$\int_1^{\infty} \frac{du}{u^2} = \lim_{a\to\infty} \int_1^a u^{-2}\,du = \lim_{a\to\infty} \left.\frac{-1}{u}\right]_{u=1}^{u=a}$

$= \lim_{a\to\infty} \left[\frac{-1}{a} + 1\right] = \boxed{1}$

#4. $\int_2^{\infty} \frac{dx}{x(\ln x)^k}$ $\left(\begin{array}{lll} u = \ln x & x=2 & u = \ln 2 \\ du = \frac{1}{x}dx & x\to\infty & u\to\infty \end{array}\right)$

$\lim_{a\to\infty} \int_{\ln 2}^{a} \frac{du}{u^k} = \lim_{a\to\infty} \left[\boxed{\frac{u^{(1-k)}}{1-k}}\right]_{u=\ln 2}^{u=a} =$

$u' \cdot u^{-k}$ $\qquad = \lim_{a\to\infty} \left[\frac{1}{(1-k)u^{k-1}}\right]_{u=\ln 2}^{u=a} = \lim_{a\to\infty} \left[\frac{1}{a^{k-1}} - \frac{1}{(\ln 2)^{k-1}}\right]$

$k-1>0$ $\quad (k>1)$ In order to converge

#5. $\int e^{-x}\cos x\,dx$

u	dv
$\cos x$ (+)	e^{-x}
$-\sin x$ (−)	$-e^{-x}$
$-\cos x$ (+)	e^{-x}

$= -\cos x e^{-x} + \sin x e^{-x} - \int \cos x \cdot e^{-x}\,dx$

a)

$2\int e^{-x}\cos x\,dx = -\cos x e^{-x} + \sin x \cdot e^{-x} + C$

$\int_0 e^{-x}\cos x\,dx = \frac{1}{2}\left[-\cos x e^{-x} + \sin x e^{-x}\right] + C$

$= \frac{1}{2}\sin x e^{-x} - \frac{1}{2}\cos x \cdot e^{-x} + C$

$\int e^{-x}\sin x\,dx$ ③

u	dv
$\sin x$ (+)	e^{-x}
$\cos x$ (−)	$-e^{-x}$
$-\sin x$ (+)	e^{-x}

$= \sin x e^{-x} - e^{-x}\cos x - \int \sin x e^{-x}\,dx$

$2\int \sin x e^{-x}\,dx = \sin x e^{-x} - \cos x e^{-x}$

$\int_0^\infty \sin x e^{-x}\,dx = -\frac{1}{2}\sin x e^{-x} - \frac{1}{2}\cos x \cdot e^{-x} + C$

$\lim_{a\to\infty}\left[-\frac{\sin a}{2e^a} - \frac{\cos a}{2e^a}\right] - \left[-\frac{\sin 0}{e^0} - \frac{\cos 0}{2}\right]$

b) $\lim_{a\to\infty}\left[\underbrace{\frac{\sin a}{2e^a}}_{0} - \underbrace{\frac{\cos a}{2e^a}}_{0}\right] - \left[\underbrace{\frac{\sin 0}{2}}_{0} - \frac{\cos 0}{2}\right] = \boxed{\frac{1}{2}}$

#6. (i) $f(x) = (\ln x)^2$

$f'(x) = \frac{2\cdot \ln x}{x}$

(ii) $g(x) = \ln(f(x))$

$g'(x) = \frac{1}{f(x)} f'(x)$

$= \frac{1}{(\ln x)^{\cancel{2}}}\left(\frac{2\cdot \cancel{\ln x}}{x}\right)$

$= \frac{2}{(\ln x)(x)}$

iii) $g'(x) = \frac{1}{(\ln x)x}$

$g'(x)$: $x=0$ (−) $x=1$ (+)

Increasing $(1,\infty)$

$x=0$, $x=1$ are Vertical Asymptotes.

#7. $u_n = \frac{3n+2}{2n-1}$ $\quad n \in \mathbb{Z}^+$

a) $\lim_{n\to\infty} \frac{(3n+2)\div n}{(2n-1)\div n} = \lim_{n\to\infty} \frac{3+\cancel{\frac{2}{n}}^{\,0}}{2-\cancel{\frac{1}{n}}_{\,0}} = \boxed{\frac{3}{2}}$

b) (i) $-0.1 < \frac{3n+2}{2n-1} - \frac{3}{2} < 0.1$ $\Rightarrow$ $\boxed{-0.1 < \frac{7}{2(2n-1)} < 0.1}$

$+1.5 \qquad\qquad +1.5$ $\qquad \Rightarrow$ $n > \frac{1}{2}\left(1+\frac{7}{2(0.1)}\right)$

$-1.4 < \frac{3n+2}{2n-1} < 1.6$ $\quad n \in \mathbb{Z}^+$

$\boxed{n = 18}$

OR Graphs: $y_1 > \frac{3n+2}{2n-1} - 1.6$

$y_2 < \frac{3n+2}{2n-1} + 1.4$

(ii) $-0.0001 < \frac{3n+2}{2n-1} - \frac{3}{2} < 0.0001$

OR Graph. $y_1 > \frac{3n+2}{2n-1} - \frac{3}{2} - 0.0001$

$y_2 < \frac{3n+2}{2n-1} - \frac{3}{2} + 0.0001$

$\boxed{n = 175000}$ $\quad$ $n > \frac{1}{2}\left(1+\frac{7}{2(0.000}\right.$

(c) $\lim_{n\to\infty} \left(\frac{3n+2}{2n-1}\right)\cdot\frac{1}{n}$

$= \lim_{n\to\infty} \frac{3n+2}{2n^2-1} \frac{\div n}{\div n} = \lim_{n\to\infty} \frac{3+\cancel{\frac{2}{n}}^{\,0}}{2n-1} = \boxed{0}$ Converges.

$$\lim_{n\to\infty} \frac{1}{2\left(\frac{3n+1}{2n-1}\right)-2} = \lim_{n\to\infty} \frac{1}{\frac{6n+2}{2n-1} - \frac{2(2n-1)}{2n-1}}$$

$$= \lim_{n\to\infty} \frac{2n-1}{6n+2-4n+2} = \lim_{n\to\infty} \frac{2n-1 \div n}{2n+4 \div n}$$

$$= \lim_{n\to\infty} \frac{2-\frac{1}{n}}{2+\frac{4}{n}} = \boxed{1}$$

$$\lim_{n\to\infty} (-1)^n \left[\frac{3n+2}{2n-1}\right] = \text{Either } \boxed{\frac{3}{2}} \text{ or } \boxed{-\frac{3}{2}}.$$