D.E IB Questions. (Work on seperate paper: use a pen)
HW for 9/27 (Friday)

1. Consider the differential equation $\frac{dy}{dx} + y\tan x = \cos^2 x$, given that $y = 2$ when $x = 0$.

(a) Use Euler's method with a step length of 0.1 to find an approximation to the value of y when $x = 0.3$. *[5 marks]*

(b) (i) Show that the integrating factor for solving the differential equation is $\sec x$.

(ii) Hence solve the differential equation, giving your answer in the form $y = f(x)$. *[10 marks]*

2. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{1+x}$, where $x > -1$ and $y = 1$ when $x = 0$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 0.5$. *[7 marks]*

(b) (i) Show that $\frac{d^2y}{dx^2} = \frac{2y^3 - y^2}{(1+x)^2}$.

(c) (i) Solve the differential equation.

(ii) Find the value of a for which $y \to \infty$ as $x \to a$. *[6 marks]*

[Maximum mark: 7]

3. Find the general solution of the differential equation $t\frac{dy}{dt} = \cos t - 2y$, for $t > 0$.

4. Solve the differential equation

$$x^2\frac{dy}{dx} = y^2 + 3xy + 2x^2$$

given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$.

5. Show that the solution of the differential equation

$$\frac{dy}{dx} = \cos x \cos^2 y,$$

given that $y = \frac{\pi}{4}$ when $x = \pi$, is $y = \arctan(1 + \sin x)$. *[5 marks]*

1B Questions

1a. $\frac{dy}{dx} = \cos^2 x - y\tan x$

$dy = (\cos^2 x - y\tan x)\,dx$

x	y	dy
0	2	0.1
0.1	2.1	0.0779
0.2	2.1779	0.0519
0.3	2.2298	
	$\boxed{2.23}$	

$dy = 0.1(\cos^2 0 - 2\tan 0)$
$= 0.1(1-0)$
$= 0.1$

1bi. $y' + (\tan x)y = \cos^2 x$

$p = \tan x$

$I(x) = e^{\int \tan x\,dx} = e^{-\ln|\cos x|} = \boxed{\sec x}$

1bii. $\sec x \cdot y' + \sec x \tan x \cdot y = \cos x$

$\frac{d}{dx}[\sec x \cdot y] = \cos x$

$\int d[\sec x \cdot y] = \int \cos x\,dx$

$\sec x \cdot y = \sin x + C$

$\boxed{y = \sin x \cos x + C\cos x}$

2a. $dy = \frac{y^2}{1+x}\,dx$

x	y	dy
0	1	0.1
0.1	1.1	0.11
0.2	1.21	0.1220
0.3	1.3320	0.1365
0.4	1.4685	0.1540
0.5	1.6225	
	$\boxed{1.62}$	

$dy = \frac{1^2}{1+0} \cdot 0.1$

2b. $\frac{d^2y}{dx^2} = \frac{(y^2)'(1+x) - (y^2)(1+x)'}{(1+x)^2}$

$= \frac{2y \cdot \frac{dy}{dx} \cdot (1+x) - y^2}{(1+x)^2}$

$= \boxed{\frac{2y^3 - y^2}{(1+x)^2}}$

2ci. $\int \frac{dy}{y^2} = \int \frac{dx}{1+x}$

$-\frac{1}{y} = \ln|1+x| + C$

$-y = \frac{1}{\ln|1+x| + C}$

$y = \frac{-1}{\ln|1+x| + C}$

$C = -1$

$\boxed{y = \frac{-1}{\ln|1+x| - 1}}$

2cii. $\boxed{\cancel{a = -1}}$ $y \to \infty$ as $\ln|1+x| - 1 \to 0$

$\ln|1+x| = 1$

$1 + x = e$

$x = e - 1$

3. $y' + \frac{2}{t} \cdot y = \frac{\cos t}{t}$

$p = \frac{2}{t}$

$I(t) = e^{\int \frac{2}{t}dt} = e^{\ln t^2} = t^2$

$t^2 y' + 2ty = t\cos t$

$\frac{d}{dt}[t^2 y] = t\cos t$

$\int d[t^2 y] = \int t\cos t\,dt$

$u = t \quad dv = \cos t$
$du = dt \quad v = \sin t$

$t^2 \cdot y = t\sin t + \cos t + C$

$\boxed{y = \frac{\sin t}{t} + \frac{\cos t}{t^2} + \frac{C}{t^2}}$

4. $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 2$

$v + x \cdot \frac{dv}{dx} = v^2 + 3v + 2$

$\frac{x}{dx} = \frac{v^2 + 2v + 2}{dv}$

$\int \frac{dx}{x} = \int \frac{dv}{(v+1)^2 + 1}$

$\ln x = \arctan\left(\frac{y}{x} + 1\right) + C$

$\ln 1 = \arctan 0 + C$

$0 = 0 + C$

$C = 0$

$\boxed{x(\tan(\ln x) - 1) = y}$

5. $\frac{dy}{dx} = \cos x \cos^2 y$

$\frac{dy}{\cos^2 y} = dx \cos x$

$\int \sec^2 y\,dy = \int \cos x\,dx$

$\tan y = \sin x + C$

$\tan \frac{\pi}{4} = \sin \pi + C$

$1 = 0 + C$

$C = 1$

$\boxed{y = \arctan(\sin x + 1)}$